The Fourth Root of 2,741,583,974

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On July 30, 1690 the Royal Society of London met to discuss a wide and wonderful variety of scientific topics. In the official minutes of that meeting, Sir Hans Sloane (1660–1753) noted that Robert Hooke (1635–1703) advanced the proposition that the "flammable materialls within the bewells of the Earth do gradually burn away and consume and are not again repared" which led him to conjecture that earthquakes and volcanic eruptions will decrease in intensity as the Earth grows older and cools. Dr. Edward Tyson (1651–1708) displayed an "extraordinary stone out of the bladder of an ancient man" which he used to demonstrate how the stone grew over time by "accression of new matter" [Sloane, n.d.]. After more discussion on the causes of earthquakes, Edmund Halley (1656–1742) took the floor. Sloane recorded that [Sloane, n.d.]:

> Halley related that Mr. Raphson had invented a method of solving all sorts of equations, and giving their roots in infinite series, which converge apace, and that he had desired of him an equation of the fifth power to be proposed to him, to which he returned an answer true to seaven figures in much less time than it could have been effected by the Known methods of Vieta.[1]

Meetings of the Royal Society in the late 17th century often featured this mix of presentations from disparate scientific fields. Indeed, in their previous meeting on July 23rd, the fellows watched a demonstration in which a candle was shot out of a musket and penetrated a board 0.7 inches thick, discussed a recent earthquake in the Leeward Islands, and read a paper from Jacob Bernoulli (1655–1705) on the latus rectum of conic sections.[2] The variety of topics discussed in such meetings might surprise modern readers who are accustomed to thinking of scientific disciplines (e.g., anatomy, physics, geology, mathematics) as distinct fields, but the separation of the sciences into their individual silos was still in the future when the Royal Society first heard of Joseph Raphson (1648–1715) and his extraordinary method for solving equations.

Task 1 Describe two ways in which the separation of science into distinct branches is beneficial, and two ways in which such separation could be problematic.

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[1] "Vieta" refers to François Viète (1540–1603), a French algebraist whose work on numerical solutions of algebraic equations, *De numerosa potestatum* (ca. 1600), became influential in the subsequent generation of European mathematicians. Viète's methods for extracting roots can be traced to the work of 12th-century Arabic mathematicians including Sharaf al-Din al-Tusi (1135–1213). For an in-depth survey of the historical development of the Newton-Raphson method, see *Historical Development of the Newton Raphson Method* [Ypma, 1995].

[2] The *latus rectum* of a conic section is the chord which passes through the focus of the curve, and is parallel to the directrix.

Joseph Raphson was an English mathematician. In 1690, he published *Analysis æquationum universalis* (*Universal analysis of equations*)[3] in which he introduced a method for approximating the roots of a given equation. Finding solutions to equations is the bread-and-butter of mathematical progress, but not all equations can be solved directly. Most of us have memorized a method for solving quadratic equations (can you still remember the quadratic formula?), and there are algebraic methods for solving cubic and quartic (fourth-degree) polynomial equations. These were known in Raphson's lifetime, although no one had yet been able to figure out how to find roots of a general fifth-degree polynomial.[4] On page 7 of his book, Raphson demonstrated the power of his method by computing the fourth root of 2,741,583,974 to an astounding accuracy (for the time) of seven decimal places [Raphson, 1690, p.7]. Today, this technique is sometimes known as "Newton's method", or less commonly as the "Newton-Raphson method". As we shall see later it might be more accurate to refer to it as the Newton-Raphson-Simpson (NRS) method. In his *Method of Fluxions*, Isaac Newton (1643–1727) developed a similar method, and demonstrated it by approximating solutions to $x^3 - 2x - 5 = 0$ on the interval from 2 to 3 [Newton, 1736]. Although we now know that Newton wrote this work in 1669, it was not published until 1736, which means that Raphson published his method nearly 50 years before Newton published his. Although the two mathematicians seem to have developed their methods independently, their paths crossed in other ways. Raphson wrote *A History of Fluxions*, published posthumously in 1715, in which he supported the priority claims of Newton over those of Gottfried Leibniz (1646–1716) with respect to the invention of calculus [Raphson, 1715]. Newton added, as an appendix to Raphson's *History*, several letters that had been exchanged between himself and Leibniz, carefully selected so as to support Newton's priority claims.

The method presented by Raphson is a powerful technique for quickly approximating solutions to equations with a very high degree of accuracy. In Section 1 of this project, we will read passages from Raphson's *Analysis æquationum* to see how his version of the method works and gain some practice with it. In Section 2, we will then take a look at why it works from a modern calculus perspective. In Section 3, we will look at applications of this method to demonstrate its accuracy and power, including application to non-polynomial cases. In Section 4, we shall explore some coding exercises that combine the already powerful technique of the Newton-Raphson method with the computational ability of today's modern computing devices.

1 Raphson's First Proposition

On page 4 of the *Analysis æquationum*, Raphson proposed the following method for approximating the cube root of a number [Raphson, 1690, p.4]. The process is iterative, meaning that each time you will obtain a more accurate estimate of the root, which can be fed back into the procedure to

[3]The expanded title is *Analysis æquationum universalis, seu, Ad æquationes algebraicas resolvendas methodus generalis, et expedita, ex nova infinitarum serierum doctrina deducta ac demonstrata* (*Universal analysis of equations, or, A general and expedient method for resolving algebraic equations, derived and demonstrated from the new doctrine of infinite series*). Available at `https://www.proquest.com/docview/2240874742?&imgSeq=1`

[4]There is a very good reason for this. In the early half of the 19th century two mathematicians, Niels Heinrich Abel (1802–1829) and Évariste Galois (1811–1832) independently proved that no such general solution is possible, while Galois also showed that no general solution can be found for equations of degree 5 or higher.

obtain an even better estimate, and so on. The following passages[5] are taken from the 1690 edition.[6]

Proposition 1: It is proposed that $aaa = b$.

Take g to be any quantity greater than a. I say that the next g (by our method) is always smaller than the previous one, but greater than a, and therefore converges to the truth.[7]

From the hypothesis, $g - z = a$ [for some positive quantity z] and

$$ggg - 3ggz + 3gzz - zzz = aaa = b.$$

Then $-3ggz + 3gzz - zzz = b - ggg$. Therefore:

$$-z + \frac{3gzz - zzz}{3gg} = (\frac{b - ggg}{3gg} = -x),$$

or the convergent term.[8] Therefore also

$$-x = -z + \frac{3gzz - zzz}{3gg};$$

when added to g, this will give $g - z + \frac{3gzz-zzz}{3gg} = g - x = g_2$. Consequently g_2 is smaller than the previous one. But also

$$a + \frac{3gzz - zzz}{3gg} = g_2;$$

therefore g_2 is greater than a. Q.E.D.

. . . .

The same procedure must be followed in all cases.

Task 2 Write two questions, and two comments, that you have about what Raphson has written.

Let's see if we can follow Raphson's argument. To begin, he proposed that we would like to find a value a such that $a^3 = b$. We are instructed to first make an initial estimate of the value, which we call g, which differs from the true value of a by some unknown amount, which we call z. Of course, we can't know how far this initial estimate g might be off from the true value of a, but Raphson gave us a process for continually refining our estimate.

[5] All translations of the Raphson excerpts in this section were prepared by Abe Edwards, Michigan State University, 2023.

[6] A second edition of the *Analysis æquationum* appeared in 1697. It is available at https://archive.org/details/bub_gb_4nlbAAAAQAAJ.

[7] It is worth noting that, as stated here, Raphson's claim is not entirely valid, since a sequence of numbers greater than a can decrease, but not converge to a.

[8] Raphson used the phrase *Theoremati convergenti*, which can be loosely translated as "the convergent part" of the theorem. It refers to the amount added to, or subtracted from, one's initial guess in order to produce a next, more accurate approximation to the solution.

Task 3 Let's work through the details of Raphson's first proposition. He proposed a method for finding successive approximations to the number a such that $a^3 = b$. Implicit in his statements is that both a and b are positive. Raphson attempted to show that by starting with a guess g that is greater than a, then we can find a second guess g_2 that is better than the original guess.

(a) Show that if $g - z = a$, then $g^3 - 3g^2z + 3gz^2 - z^3 = b$.

(b) Verify the algebra: Show that $-z + \frac{3gz^2 - z^3}{3g^2} = \frac{b-g^3}{3g^2}$.

After this calculation, Raphson defines $-x$ to be the number $(b-g^3)/(3g^2)$. Later in the project, we'll see why it is sensible to focus on this quantity, the "convergent term."

(c) Is x positive or negative? Explain your reasoning.

Raphson next defined g_2 to be equal to $g-x$ and sets out to prove that (1) $g > g_2$ and (2) $g_2 > a$.

(d) Re-read Raphson's argument for why $g > g_2$. Explain why this inequality is true.

(e) Re-read Raphson's argument for why $g_2 > a$. Is it true that $3gz^2 - z^3 > 0$? Explain your reasoning. Hint: Write $3gz^2 - z^3 = 2gz^2 + gz^2 - z^3$ and factor out z^2 from the last two terms.

(f) Explain how this method can be iterated to produce $g_3, g_4, \ldots$ such that

$$a < \cdots < g_4 < g_3 < g_2 < g.$$

(g) These guesses, as Raphson claimed in the beginning of his proof, must converge to the true value of a. Do you agree with his logic? Why or why not?

Raphson's use of phrases such as "the next g" and "always smaller than the previous one" imply that we are to repeat this process, each time starting with the most recent value for g and obtaining an even more precise estimate which gets fed back into the algorithm as many times as we like, until the value of g "converges to the truth." Indeed, Raphson posed the following problem in order to demonstrate his method:[9]

Problem II: To extract the side of the cube from 37945.

The equation $aaa = b$.

Number: $aaa = 37945$

$x = \frac{b-ggg}{3gg}$

...

$33 = g$

[9]We have made a minor change in this passage to Raphson's original notation for the sake of consistency.

Task 4 In Problem II of his book, Raphson iterated three times, obtaining a final value of

$$g_4 = 33.60352617943808$$

as an estimate for the value of a in the given equation. Let's see if we can obtain this same result by following his method.

(a) We begin by setting $g = 33$ and $b = 37945$, as shown in the excerpt above. Explain how we know that $a > g$.

(b) Since $a > g$ and we want our guesses to converge to a, the next estimate g_2 should be larger than g. To find the value of g_2, Raphson therefore computed $g_2 = g + x$. Do this yourself using the formula for x from Raphson's Problem II. Note that you'll need to decide how many decimal digits of x you want to keep when computing g_2. Raphson only kept the first two digits past the decimal point, giving him a result of $g_2 = 33.61$. Confirm this result.

(c) Next, take your value of g_2 and use it, along with the formula for x in Raphson's Problem II, to find a value for $g_3 = g_2 + x$. Here, x will be negative, so that $g_3 < g_2$. We are now back to the case that Raphson considered in his Proposition 1, in which all the remaining estimates will decrease ever closer to the true value. In doing this step, Raphson kept the first six digits past the decimal point for x.

(d) Using your new estimate, g_3, run the same procedure once more to obtain g_4. Does your value match that of Raphson?

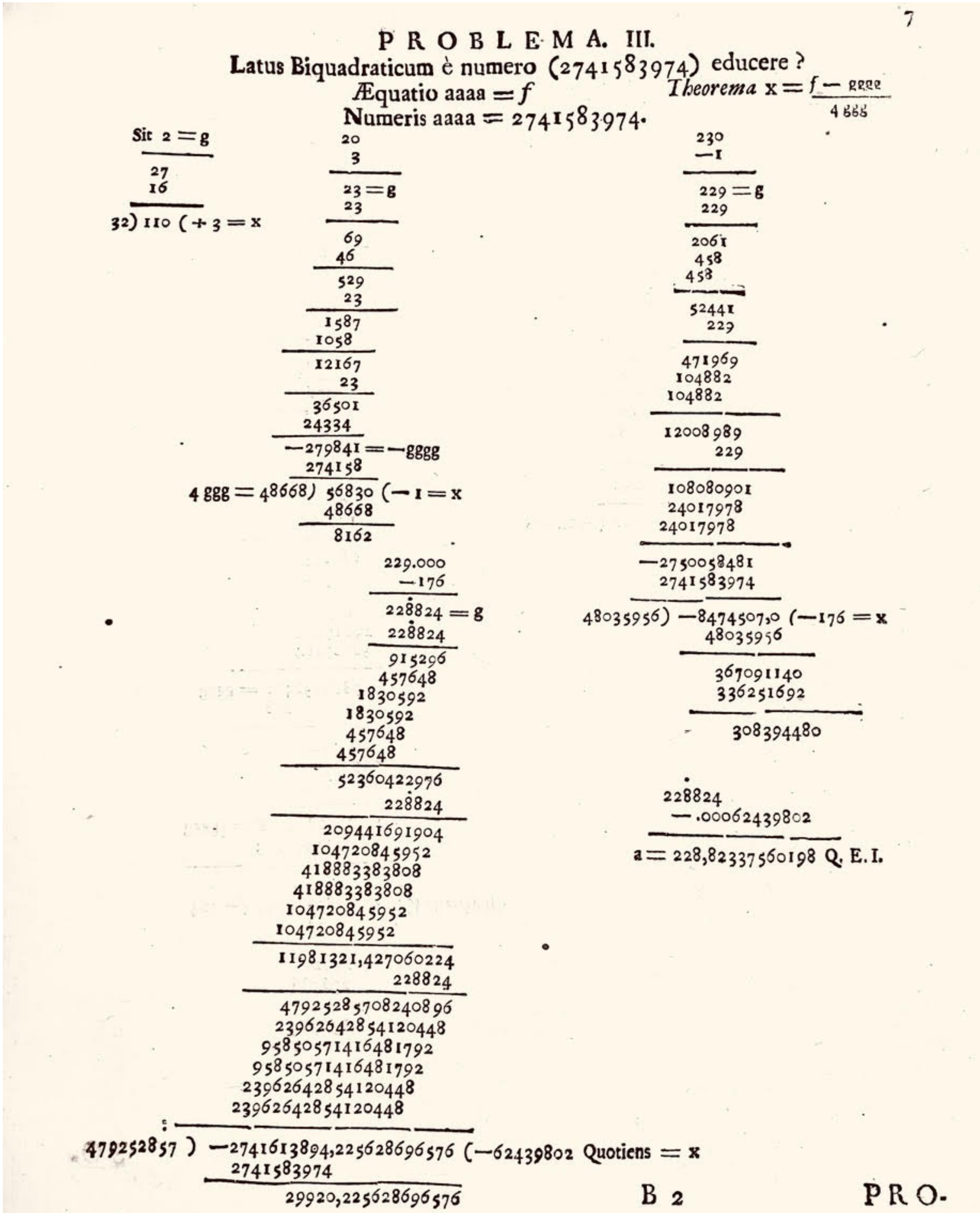
(e) Use a calculator or computer to determine the cube root of 37945. To how many digits is Raphson's estimate accurate?

(f) Eighteenth-century mathematicians, of course, had no calculator against which they could check their results. In the absence of such technology, what would a statement such as "accurate to seven decimal places" mean?

After explaining the procedure for extracting cube roots, Raphson added the following sentence:

Many other things may be deduced ... from this method which, however, remain to others. It is enough to have led, to have shown the way, if I am not mistaken, quite amply.

Nevertheless, Raphson couldn't quite seem to resist showing off the power of his method, and on page 7 of the *Analysis æquationum* he used it to find the fourth root of $2,741,583,974$. His solution is shown in Figure 1 on the following page: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PROBLEMA. III.

Latus Biquadraticum è numero (2741583974) educere ?

Æquatio aaaa = f *Theorema* x = f — gggg / 4 ggg

Numeris aaaa = 2741583974.

Sit 2 = g
27
16
32) 110 (+ 3 = x

20
3
23 = g
23
69
46
529
23
1587
1058
12167
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36501
24334
—279841 = —gggg
274158
4 ggg = 48668) 56830 (— 1 = x
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8162

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308394480

228824
— .00062439802
a = 228,82337560198 Q. E. I.

B 2 PRO-

Figure 1: Page 7 of Raphson's *Analysis æquationum*. Image courtesy of Abe Edwards.

Task 5 There are numerous features of Raphson's work in Problem III that are worthy of our attention.

(a) What do you think the Latin term "Latus Biquadraticum" means in the statement of the problem?

(b) What starting value for g did Raphson choose?

(c) In modern notation, what was Raphson's formula for x in the case of finding a fourth root?

(d) Write down two additional observations you have about the above excerpt.

One other thing we might notice upon looking at Raphson's work on this problem is that he didn't begin by directly working with all the digits of his original ten-digit number. This is because he had worked out an elegant way to manage calculations with very large numbers. Although we have the benefit of powerful computational tools, it is well worth getting a sense of how Raphson managed to perform such extraordinary feats of arithmetic.

Task 6 In this task, we'll work through the details of Raphson's Problem III that is shown in its original Latin in Figure 1.

(a) Begin by setting $g = 2$ and $f = 27$, and use $x = \dfrac{f - g^4}{4g^3}$, and combine x with g to obtain a new value, g_2. What do you obtain? How does your result seem to compare to what Raphson got for his?

(b) Raphson then used $g_2 = 23$ as his new "guess", and this time he used $f = 274158$ (four more digits than before). Explain how we might justify this step, taking your results from part (a) into account. Hint: Begin by noting that $(2.3 \times 10)^4 = (2.3)^4 \times 10^4$.

(c) With the values of g_2 and f from part (b), run Raphson's method again. What do you obtain for g_3? Does it match what Raphson obtained?

(d) As in the previous step, multiply your value of g_3 by 10, and use it as your initial guess for the fourth root of f, now with four more digits. What value do you obtain for g_4?

(e) Finally, use your value of g_4 (no need to multiply by ten now), and f to obtain an even more accurate estimate, g_5. How closely does your final result match Raphson's? How might you account for any discrepancies?

(f) Why might Raphson have chosen to approach the fourth root of f in this way (i.e., by increasing the number of digits of f in each iteration)? Hint: Instead of using your calculator, try working out one of the values of x doing the long division by hand.

Having seen that Raphson's method can quickly produce solutions of astonishing numerical accuracy, it is natural to ask: Why does the method work? In the next section, we'll look at this question from a modern point of view.

2 The Methods from a Modern Perspective

To see what's going on in Raphson's work, it's helpful to examine the terms which appear in Raphson's computation of what he called "the convergent term" (the amount added to, or subtracted from, one's initial guess in order to obtain a more precise estimate of the solution).

Task 7 In his work on the cube root of 37945, Raphson let $x = \dfrac{37945 - ggg}{3gg}$. In modern notation, we would write $x = \dfrac{37945 - g^3}{3g^2}$. What mathematical relationship do you notice between g^3 and $3g^2$? In Raphson's work involving the fourth root, do you see the same relationship between the numerator and denominator of x? Explain.

It might seem strange to suddenly see the notion of the derivative appearing in Raphson's work, and even more strange that he never referred to differentiation. Throughout his *Analysis*, Raphson proceeded in a purely algebraic way rather than using the rules of calculus. It seems that he did not associate calculus with his iterative technique. For example, in Problem IX of his book, Raphson wrote:

Problem IX.

It is proposed that $aaa - ba = c$. Equation of the second form.

$$aaa - 2a = 5$$

$$\text{Theor. } x = \frac{c + bg - ggg}{3gg - b}$$

That is, for an equation of the form $a^3 - ba - c = 0$, where the variable is a, if g is an estimate of the solution, then a better estimate can be obtained as $g + x$ where $x = \dfrac{c - (g^3 - bg)}{3g^2 - b}$. Again, we see the appearance of an expression $g^3 - bg$ and its derivative, $3g^2 - b$.

Let's see if we can write Raphson's procedure using a modern function notation.

Task 8 Suppose we would like to find a solution to $f(x) = 0$. We begin with an initial guess x_0. Using this notation, what is Raphson's formula for the next "better guess" x_1?

In modern notation, the algorithm for estimating solutions to $f(x) = 0$ is as follows. If $f(x) = 0$ has a root r and if x_i is an approximation of r, and $f'(x) \neq 0$, then the next approximation is given by:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

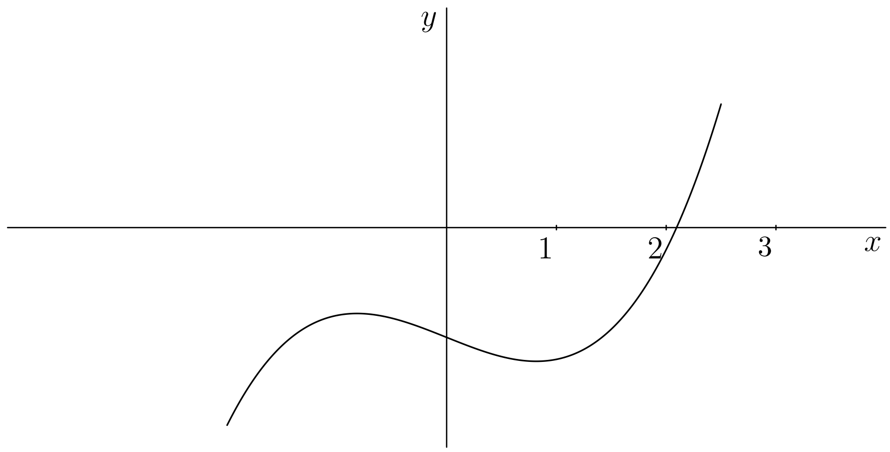
If the numbers x_i become closer and closer to r over successive iterations, then we can write

$$\lim_{n \to \infty} x_n = r.$$

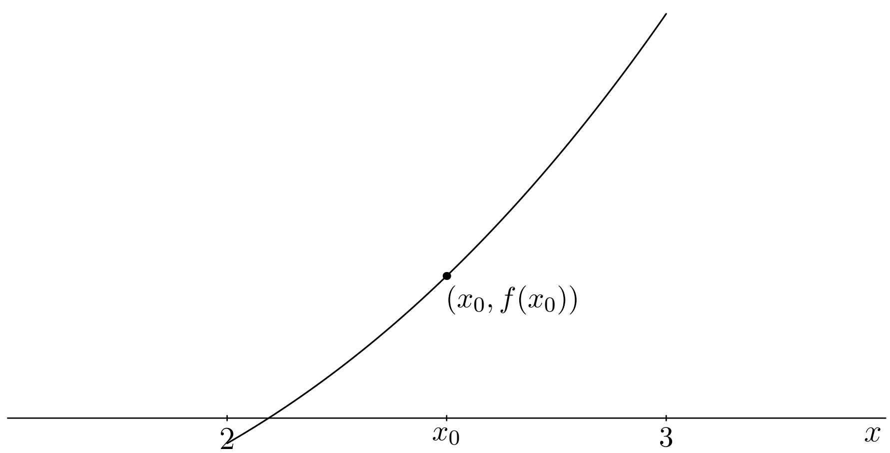
Task 9 Explain why Raphson's formula is essentially identical to the modern notation formula given above.

Today, the iteration $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ is known as the "Newton-Raphson-Simpson method" (or NRS for short), for reasons that we will examine in the conclusion of this project. In particular, we will look at the contribution made by Thomas Simpson (1710–1761), whose name is associated with the method, along with those of Raphson and Newton.

To understand why derivatives appear in the Newton-Raphson-Simpson method, let us consider Raphson's "Problem IX" from a geometric perspective. A portion of the graph of $x^3 - 2x - 5 = 0$ is shown in the following figure. We would like to estimate its root, which appears to be between 2 and 3.



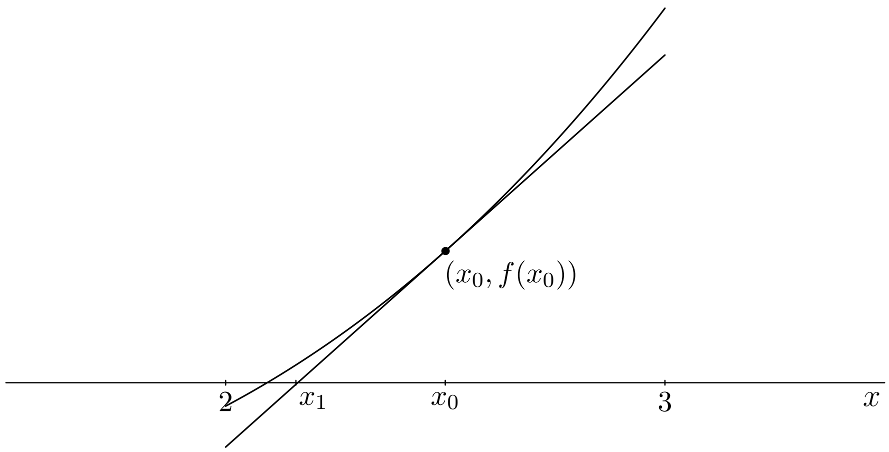
Underlying Raphson's approach is the following: Suppose we make an initial guess x_0 for the value of the root we seek. Assuming we didn't successfully guess the root on our first try, we could find the corresponding point on the curve $(x_0, f(x_0))$, which would either lie above or below the $x-$axis. For example, suppose our initial guess x_0 is to the right of the true solution. The corresponding point on the curve is shown below:



Task 10 The key idea in the NRS method is the construction of a tangent line to the curve $f(x)$ at the point $(x_0, f(x_0))$. We continue to think about $x^3 - 2x - 5 = 0$ so that $f(x) = x^3 - 2x - 5$.

(a) Let your initial guess be $x_0 = 2.5$. Find the coordinates of the point $(x_0, f(x_0))$, and then find the equation of the line tangent to $f(x)$ at that point.

(b) The portion of the curve, with the corresponding tangent line, is shown below. This tangent line is a good linear approximation to $f(x)$ near x_0, so our next guess x_1 is the point where the tangent line intersects the $x-$axis.



We then proceed using the same method, and each time our result is closer to the true solution we seek. Find the coordinates of the point $(x_1, f(x_1))$. Determine the equation of the tangent line to the curve at that point, and find the x-intercept of that tangent line.

(c) As you have seen, the key calculation in each step of this process is finding where the tangent line to $y = f(x)$ at the point (x_0, y_0) intersects the $x-$axis. Let's see if we can develop a general equation for finding those points. Recall that, given the slope m of a line, and a point (x_0, y_0) on it, the line has equation $y = m(x - x_0) + y_0$. In our scenario, the line has slope $f'(x_0)$ and passes through (x_0, y_0), so has the equation $y = f'(x_0)(x - x_0) + y_0$. Set $y = 0$ in this equation and solve for the $x-$intercept of the tangent line.

3 Further Investigations of the Newton-Raphson-Simpson Method

The Newton-Raphson-Simpson (NRS) method is *usually* very effective, and the accuracy of each iteration increases rapidly.

Task 11 As a demonstration of its power, let's approximate some well-known constants.

(a) Find an approximation to π using three iterations of the NRS method to solve $\sin x = 0$, starting from $x_0 = 3$. To how many decimal places is the approximation accurate?

(b) Find an approximation to $\sqrt{2}$ using four iterations of the NRS method to approximate a solution to $x^2 - 2 = 0$, starting from $x_0 = 2$. To how many decimal places is the approximation accurate?

Task 12 When employing the NRS method, the number of decimal places of accuracy approximately doubles with each iteration. This fact was also noted in the minutes from the July 30th, 1690 meeting of the Royal Society. Let's investigate this doubling of the accuracy in the above approximation of $\sqrt{2}$.

(a) Begin by showing that $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$.

(b) Now, show that if $y_n = x_n - \sqrt{2}$, then $y_{n+1} = \dfrac{{y_n}^2}{2(\sqrt{2} + y_n)}$.

(c) Show that if $0 < y_n < 10^{-k}$ for some positive integer k, then $0 < y_{n+1} < 10^{-2k}$.

(d) Finally, show that, for $n \geq 2$, if x_n differs from $\sqrt{2}$ by less than 10^{-k}, then x_{n+1} differs from $\sqrt{2}$ by less than 10^{-2k}.

It should be noted that, while the NRS method can give amazingly accurate approximations to a solution, there are occasions when it fails. As you complete the next tasks, start thinking about why the method doesn't work for these problems!

Task 13 Try to approximate the solution to $\sin x = 0$ starting from $x_0 = \frac{\pi}{2}$. Why does the NRS method fail? (*Hint:* What kind of point is located on $y = \sin x$ at $x = \frac{\pi}{2}$?)

Task 14 Try to approximate a solution to $-x^3 + 4x^2 - 2x + 2 = 0$ from $x_0 = 0$. What happens after a few iterations of the NRS method?

Task 15 Try to approximate a solution to $\arctan x = 0$ starting from $x_0 = 3$. What happens after a few iterations of the NRS method? Make a simple sketch of $y = \arctan x$, along with your values for $x_0, x_1, \ldots$, showing the respective tangent lines. Explain how your sketch depicts a weakness of the NRS method.

4 Coding the NRS Method

Methods of numerical approximation, such as the ones proposed by Newton, Raphson, and Simpson, are not merely artifacts from a pre-computational age. In fact, such methods are particularly well suited to computer programming applications. Their iterative nature, combined with some fairly heavy arithmetical computations, make them ideal candidates for coding exercises. While we should be impressed with both the accuracy and tenacity of 17th-century mathematicians to carry out such lengthy computations by hand, we can employ some 21st-century computational power to continue our investigation of the NRS method.

To engage with a set of coding exercises, hosted on Google Colab, based on the Newton-Raphson-Simpson method, please use the following link to a Coding Supplement:

https://colab.research.google.com/drive/1wWsDx-JFnZVjCvHjOK4y4wN3fsWYZ3Fb?usp=sharing

5 Conclusion: "Simpson, eh?"[10]

What is now known as Newton's Method or the Newton-Raphson Method did not appear in its present form, using the quotient $f(x)/f'(x)$, until Thomas Simpson's *Essays on Several Curious and Useful Subjects in Speculative and Mix'd Mathematicks* appeared in 1740. Simpson was the first to formulate the familiar iterative root finding method in terms of derivatives or, more precisely, in terms of *fluxions*, which are essentially derivatives of a variable with respect to a varying quantity such as time [Ypma, 1995, p.548]. Newton, circa 1669,[11] described his method to find successive

[10] A well-known expression of Homer Simpson's employer, Mr. Burns, on the animated television program *The Simpsons*.

[11] Historians believe that *De analysi* was written in 1669; the contents of *De analysi* was not published until it was included in *De methodis fluxionum et serierum infinitorum* in 1736 [Kollerstrom, 1992, p.347].

approximations to roots of a polynomial equation in *De analysi per æquationes numero terminorum infinitas* (*On Analysis by means of equations of an infinite number of terms*). Kollerstrom argues that Newton's original method was neither an iterative method nor a method that used differential calculus [Kollerstrom, 1992].

Raphson's method is an iterative method and, as we have illustrated, in his *Analysis æquationum* he worked out many examples to a high degree of accuracy. However, Raphson's methods do not use differential calculus either. Both Newton and Raphson's methods were algebraic and relied on dropping higher order terms of polynomials to produce corrections to the original guess. Kollerstrom implies that Raphson's method appears to be a direct modification of Newton's method, which had been shared among mathematicians at this time in England. Contemporaries preferred Raphson's method as it involved fewer or simpler computations. On the other hand, Simpson's method is explicitly formulated as an iterative method using the quotient of the original function by its derivative (or quotient of fluxions). According to Ypma, the equation using the notation $f'(x)$ appeared in Lagrange's *Traité de la Résolution des Équations Numériques*, published in 1798 [Ypma, 1995, p.549].[12]

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[12]For more on the history of this topic, we refer the interested reader to Kollerstrom's Ypma's highly engaging respective accounts. Ypma's article also includes an extensive bibliography

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is intended for use in a first-semester calculus course, after students have been introduced to the standard differentiation techniques. Students are introduced to the Newton-Raphson method (as it is usually called) of approximating roots of polynomials. Rather than simply presenting the method as a recipe, the project positions the development of the method within the broader context of late-17th century scientific discovery. The project also includes tasks which are intended to give students practice with writing and implementing routines in the Python programming language.

This project may also be used in a history of mathematics course for students who have already taken a calculus course.

Student Prerequisites

Students should have learned the concept of a derivative, be familiar with the standard differentiation techniques, and understand the interpretation of the derivative as the slope of a tangent line.

PSP Design and Task Commentary

In the first section of this project, students read and interpret selections from Joseph Raphson's *Analysis æquationum* in order to develop an iterative method (today known as the Newton-Raphson method) for approximating roots of polynomials. After verifying Raphson's algebra in Task 3, students use Raphson's method to extract the cube root of 37945 and compare the accuracy of Raphson's method with the value obtained by a computer. The final question in Task 4 is meant to spark discussion about how one could know that their estimate was accurate to a given level of precision, without knowing the value of the root ahead of time! For Task 5, the instructor may wish to prepare a word processed or typeset version of the first few lines of text in Figure 1 for those that require the use of screen readers. In Task 6, students use Raphson's method to extract the fourth root of a ten-digit number but it is worth noting that Raphson alters his method slightly in order to make the long division steps easier.

In the second section, students work through Problem 9 of the *Analysis æquationum*, using a modern approach in order to see how successive tangent lines converge on the polynomial root. This should help them see why the NRS method works from the perspective of modern calculus. In the third section, students are taken through several tasks in which they investigate how well the method works, and under which conditions it might fail! In Task 13, for example, students use the method to find decimal approximations for π and $\sqrt{2}$. Students should find that the number of correct digits roughly doubles with each iteration, and in Task 14, students investigate this behavior more formally. The final tasks of this section are meant to demonstrate scenarios in which NRS might fail to provide an accurate approximation, and students should be able to articulate some limitations of this method.

In the fourth section of the PSP, students are introduced to the Python programming language and are guided through a series of coding tasks in which they use NRS to approximate solutions to the same polynomials Raphson was working on in the 1690s.

Suggestions for Classroom Implementation

It would be ideal for students to encounter this project in lieu of the instructor's normal textbook exposition of Newton's method, and/or in the context of other applications of differentiation such as optimization. If using this project in a history of mathematics course, in which students have already had calculus, instructors have more flexibility with respect to timing. The project authors wrote this PSP for use in their own first-semester calculus classes. Students should expect to do some of the project before class, and collaborate on many tasks during class. Instructors who have less experience with the Python programming language may wish to work through Section 3 on their own before leading students through it, although the tasks have been designed for novice coders!

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on three 50-minute class periods)

This suggested implementation schedule is based on three 50-minute periods. Instructors who wish to run the PSP in less time may wish to follow the alternative implementation schedule below. Regardless of the duration of the project, instructors should impress upon, and potentially incentivize, advance reading and homework to be completed before the start of class. The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students.

Day Zero (preparation and homework). Prior to the start of in-class implementation, the instructor should assign reading the PSP from the opening page through Task 3. Students should be asked to write up complete responses to Tasks 1 through 3 and be prepared to share their results with a small group. Tasks 1 and 2 are open-ended and meant to spark discussion, but students should all obtain the same solution to Task 3.

Day One. At the start of class, have students form groups of 3–4 and briefly share their responses to Tasks 1 through 3. They should spend the next 45 minutes carefully working through the PSP with guided help from the instructor, with the goal of finishing Section 1. Note that in Task 6, Raphson changes his process slightly, and students may need some assistance justifying and implementing the revised method. If time permits, have students work through Tasks 7 and 8 in class. Otherwise Tasks 7, 8, and 9 may be assigned as homework.

Day Two. Upon returning to class, go over Task 7, in which it is crucial for students to articulate the connection between the derivative and the method of approximating roots. After Task 8 the students are shown the modern version of Newton-Raphson-Simpson. Devote the next 20-30 minutes to having students work through the rest of Sections 2 and 3. Note that Task 12 is challenging, and might be best considered as optional, although stronger students may appreciate the opportunity to go deeper with these ideas! If time is short, Tasks 13 through 15 could be assigned as homework, but at the very least, it is important for students to understand that there are situations in which Newton-Raphson-Simpson might fail to produce a sequence that converges on the desired root.

Day Three (Implementing the Coding Supplement). In our conceptualization of the project, students would work on the Coding Exercises as the final culmination of their work. Allow at least 30 minutes for students to work through the supplement, and more time if students have no prior experience with coding. This could be assigned as homework, but there is value in having the students collaborate on the coding tasks. We suggest prioritizing this section over Tasks 12–15, if necessary.

Alternative Implementation Schedule (based on two 50-minute class periods)

We understand that some instructors may only have two days (at most) to spend on this project. In such circumstances, we suggest the following modifications to the implementation schedule given above:

Days Zero and One. Same as above, with the goal of having students finish Task 6 by the end of the first full day of instruction. Assign Tasks 7, 8, and 9 as homework.

Day Two. Begin with a brief discussion of Tasks 7, 8, and 9. Have students work through Tasks 10 an 11. Use the remaining class time to either work on Tasks 13 through 15, or the Coding Exercises (our preference).

An Alternative Approach to the Coding Supplement

An alternative approach to this project would ask students to work on this supplement in-stream with the other project tasks. Instructors who wish to take this approach could do the following:

- After Task 4: Point students towards Sections 1 and 2 of the supplement.
- After Tasks 5 and 6: Point students towards Section 3.
- After Tasks 10-12: Ask students to write code to find the roots of the equations in these tasks to a specified precision.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in a first-semester calculus course. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). Each of these is designed to be completed in 1–2 class days. Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_calculus.

- *The Derivatives of the Sine and Cosine Functions*, Dominic Klyve
- *Fermat's Method for Finding Maxima and Minima*, Kenneth M Monks
- *L'Hopital's Rule*, Danny Otero
- *Three Hundred Years of Helping Others: Maria Agnesi on the Product Rule*, Kenneth M Monks

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