

# From Quaternions and Octaves to Circuit Design: An Exploration of Modern Boolean Algebra via its Origins in British Symbolic Algebra

November 22, 2025

While students of mathematics the world over are today accustomed to manipulating algebraic symbols according to known rules, the study of what was known as “symbolic algebra” in the nineteenth century was a distinctly British affair. During the eighteenth century, British mathematics was cut off from that of the European continent, in part due to a priority controversy concerning who deserved credit for the discovery of calculus: Gottfried Leibniz (1646–1716) or Isaac Newton (1643–1727). On the continent, the differential techniques and notation of Leibniz allowed mathematicians to make significant advances in the development of calculus. In contrast, British mathematics was nearly stagnant. By the start of the nineteenth century, however, significant changes emerged in English society as a result of the Industrial Revolution, which in turn impacted the role of universities in promoting research and science. In 1812, a group of mathematicians at Cambridge formed ‘The Analytic Society’ and began to import continental methods of symbolic manipulation into the university curriculum. As a result, certain concerns about the foundations of algebra emerged, including questions about the meaning of negative and imaginary numbers. The idea of a ‘symbolical algebra’ developed from the efforts of British mathematicians to resolve these questions and concerns.

In contrast to “arithmetical algebra,” which derives its laws from the actual meaning of operations on numbers, a “symbolical algebra” begins with formal unrestricted laws for a given set of symbols and operations, and only later interprets these as having particular meaning. In his *Trigonometry and Double Algebra* [DeMorgan, 1849, p. 94], De Morgan gave the following example of a symbolic algebra:

Given symbols  $M$ ,  $N$ ,  $+$ , and one sole relation of combination, namely that  $M + N$  is the same as  $N + M$ .

He then provided five distinct interpretations which would give meaning to this particular (commutative) algebra, ranging from

$M$  and  $N$  may be magnitudes, and  $+$  the sign of addition of the second to the first  
to

$M$  and  $N$  may be numbers, and  $+$  the sign of multiplying the first to the second  
to

$M$  and  $N$  may be nations, and  $+$  the sign of the consequent having fought a battle with the antecedent.<sup>1</sup>

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<sup>1</sup>It would seem that De Morgan may have had the Napoleonic wars — yet another cause behind the isolation of British mathematics from that of Europe in the nineteenth century — in mind when he composed this example.

Because of the nature of its initial definition in a set of formal operational laws — and again in contrast to arithmetical algebra which has all the standard algebraic properties — a symbolic algebra can have quite unusual algebraic properties. For instance, although addition in the system of *quaternions* is well-behaved, the commutativity property generally fails for multiplication. In other words, we can *not* assume that  $xy$  has the same value as  $yx$ . While today’s students of mathematics become familiar with a variety of non-commutative algebraic operations (*Can you think of such an operation?*), the non-commutativity of quaternion multiplication constituted a major breakthrough in its day, comparable to the discovery of non-Euclidean geometry. In fact, it took the creator of quaternions, Sir William Rowan Hamilton (1805–1865) six entire years of searching for an algebraic system that could be reasonably interpreted in the three-dimensional space of physics before he realized he needed to abandon commutativity in order to find a solution. The system of *octaves* (or *octonions*), another of symbolic algebra creation, is even more bizarre since multiplication is neither commutative nor associative.



Plaques commemorating Hamilton and his discovery of quaternions on Broom Bridge, Ireland.

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In this project, we explore the challenges of computing in these two early symbolic algebras, quaternions and octaves, before moving on to explore an important structure whose roots also lie in British symbolic algebra: namely, Boolean algebra. We’ll begin our examination of Boolean algebra by reading the words of its inventor, George Boole (1815–1864). Through excerpts from his 1854 *An Investigation of the Laws of Thought*, we’ll see that while the two operations of a Boolean algebra (typically denoted  $+$  and  $\cdot$ ) are well-behaved in some respects — for example, both are commutative, so that  $x + y = y + x$  and  $x \cdot y = y \cdot x$  — they nevertheless have some unexpected algebraic properties — like distributivity of addition over multiplication, so that  $x + y \cdot z = (x + y) \cdot (x + z)$ . After reading Boole, we’ll move on to look at how those laws were applied to the important problem of circuit design in the work of Claude Shannon (1916–2001). Finally, we’ll close this project with a glimpse of modern Boolean algebra as a means to consolidate our understanding of its intriguing properties.

# 1 Hamilton's System of Quaternions

Irish mathematician Sir William Rowan Hamilton (1805–1865) was a child prodigy with a gift for both languages and mathematics. His academic talents were fostered by his uncle James Hamilton, an Anglican clergyman with whom he lived from the age of 3. Under his uncle's tutelage, Hamilton mastered a large number of languages — including Latin, Greek, Hebrew, Persian, Arabic and Sanskrit — by the age of 10. His early interest in languages was soon eclipsed by his interests in mathematics and physics, spurred in part by his contact with an American calculating prodigy. Hamilton entered Trinity College in Dublin in 1823, and quickly distinguished himself. He was appointed Astronomer Royal of Ireland at the age of 22 based on his early work in optics and dynamics. Highly regarded not only by his nineteenth-century colleagues, Hamilton is today recognized as a leading mathematician and physicist. His 1843 discovery of quaternions<sup>2</sup> is remembered today as his greatest mathematical contribution.

A quaternion is a linear expression of the form  $a + bi + cj + dk$ , where  $i, j, k$  are three (independent) imaginary units and  $a, b, c, d$  are real numbers. Addition of quaternions in the usual 'componentwise' way is straightforward, as shown here:

$$(3 + 4i + 3j - 5k) + (9 - 6i + 7j + k) = 12 - 2i + 10j - 4k.$$

Multiplying quaternions is somewhat more complicated — just think about distributing multiplication over addition in order to expand a product like  $(3 + 4i + 3j - 5k)(9 - 6i + 7j + k)$ . Even after finding all 16 individual products, we still need to write the result in the form of a quaternion:  $a + bi + cj + dk$ . To do this simplification, Hamilton realized that he needed to determine the results of nine specific products:

$$i^2, ij, ik, ji, j^2, jk, ki, kj, k^2.$$

This meant that he could focus on the multiplicative properties of just four basic elements:  $1, i, j, k$ .

Today, mathematicians define the *quaternion group*<sup>3</sup> to be the set  $G$  containing these four basic elements together with their opposites, and a product on this set defined by the multiplication rules given as follows.

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<sup>2</sup>The research project that led Hamilton to the discovery of quaternions was the search for an algebraic system that could be reasonably interpreted in the three-dimensional space of physics, in a manner analogous to the interpretation of the algebra of complex numbers  $a + bi$  in a two-dimensional plane. After six years of unsuccessful work on this problem, he found a solution only by abandoning the algebraic property of commutativity. Hamilton also replaced 'triplets' by the 'four-dimensional' quaternion  $a + bi + cj + dk$ . Soon after Hamilton's discovery, physicists realized that only the 'vector part'  $bi + cj + dk$  of a quaternion was needed to represent three-dimensional space.

<sup>3</sup>A group is an algebraic structure consisting of a set of elements  $G$  and one binary operation  $(\cdot)$  on  $G$  that satisfies the following three properties:

- Multiplication is associative on  $G$ : For all  $x, y, z \in G$ ,  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .
- $G$  contains a multiplicative identity  $1$ : For all  $x \in G$ ,  $x \cdot 1 = 1 \cdot x = x$ .
- $G$  satisfies the inverse property: For all  $x \in G$ , there is some  $y$  in  $G$  such that  $x \cdot y = 1 = y \cdot x$ .

Interestingly, the article that is considered by many to be the first paper on the abstract group concept was also written by someone steeped in the workings of symbolic algebra, the British mathematician Arthur Cayley (1821–1895). See the student project [Barnett, 2011] for an introduction to groups via primary sources, including Cayley's landmark 1854 paper.

$$\begin{aligned}
G &= \{1, -1, i, -i, j, -j, k, -k\} \\
i^2 = j^2 = k^2 &= -1 & (-1)i = i(-1) &= -i \\
(-1)^2 &= 1^2 = 1 & (-1)j = j(-1) &= -j \\
ijk = i(jk) = (ij)k &= -1 & (-1)k = k(-1) &= -k
\end{aligned}$$

Two additional assumptions are also made as part of the definition of a group (see footnote 3):

- Multiplication is associative: For all  $x, y, z$  in  $G$ ,  $(xy)z = x(yz)$ .
- 1 is a multiplication identity: For all  $x$  in  $G$ ,  $1x = x = x1$ .

In the tasks below, you will use these rules to find all 64 products of the quaternion group. As an example of how to compute other products, here is a proof that  $ij = k$ . We begin by multiplying both sides of the equation ‘ $ijk = -1$ ’ on the right by  $-k$ , and then simplifying according to the given rules as follows:

$$\begin{aligned}
ijk = -1 &\Rightarrow (ijk)(-k) = (-1)(-k) \\
&\Rightarrow (ij)k[(k)(-1)] = (-1)[(-1)k] \\
&\Rightarrow (ij)k^2(-1) = (-1)^2k \\
&\Rightarrow (ij)[(-1)(-1)] = (1)k \\
&\Rightarrow ij(1) = k \\
&\Rightarrow ij = k
\end{aligned}$$

**Task 1** Go back through the example above and underline all the places where associativity was used in the computation. How many instances did you find in all?

**Task 2** Use a similar strategy to the example above ( $ij = k$ ) to show that  $jk = i$  and  $ki = j$ .

Be careful not to commute elements unless the given rules explicitly allow this!  
(For example,  $(-1)k = k(-1) = -k$  allows us to commute  $k$  and  $-1$ .)

$$jk = i$$

$$ki = j$$

**Task 3** From the preceding tasks, we know that  $j = ki$ ,  $i = jk$  and  $k = ij$ .

Use these three facts<sup>4</sup> to show that  $ji = -k$ ,  $kj = -i$  and  $ik = -j$ .

Explain how this shows that quaternion multiplication is not commutative.

**Task 4** What are the values of  $(-i)(-j)$  and  $(-i)j$ ? Use associativity to justify your answers.

**Task 5** This task examines the multiplication table for the quaternion group  $G = \{1, i, j, k, -1, -i, -j, k\}$ .

- (a) Use the table below to construct an  $8 \times 8$  multiplication table for  $G$  using the multiplication facts given or derived thus far, along with any additional derivations needed to fill in all 64 products. **Follow the convention of “row  $\times$  column”**, so that row headings give the left factor and column headings the right factor of each product.

	1	$i$	$j$	$k$	$-1$	$-i$	$-j$	$-k$
1	1	$i$	$j$	$k$	$-1$	$-i$	$-j$	$-k$
$i$	$i$	$-1$	$k$	$-j$	$-i$			
$j$	$j$	$-k$	$-1$	$i$	$-j$			
$k$	$k$	$j$	$-i$	$-1$	$-k$			
$-1$	$-1$	$-i$	$-j$	$-k$	1			
$-i$	$-i$							
$-j$	$-j$							
$-k$	$-k$							

- (b) What patterns do you see in the completed table?

## 2 Octaves

The discovery of an algebraic system involving 8 elements  $(1, i, j, k, p, q, r, s)$  and their opposites was discovered independently by two men soon after Hamilton’s announcement of the quaternions. One of these men was the British mathematician Arthur Cayley (1821–1895), who is known for writing the first paper on symbolics groups (see footnote 3). The other was the Irish jurist and mathematician John T. Graves (1806–1870), a close friend of Hamilton.<sup>5</sup> Concerning the properties of octaves, Hamilton declared in his book *Lectures on Quaternions*:

To this associative principle, or property of multiplication, I attach great importance. . . . . The *absence* of associativity appears to me to be a great inconvenience in the octaves or octonomials of Messrs. J. T. Graves and Arthur Cayley. [Hamilton, 1853, p. 61]

To see how this inconvenience arises, consider the following (partial)<sup>6</sup> multiplication table for the system of octaves:

<sup>4</sup>If you have studied multivariable calculus, you may be familiar with the so-called “right-hand rule” as a way to remember the three products  $j = ki$ ,  $i = jk$  and  $k = ij$ . If not, ask your instructor!

<sup>5</sup>Graves appears to have had precedence in terms of discovery, but Cayley was the first to publish. For this reason, the elements of this system of octaves are sometimes referred to as ‘Cayley numbers.’

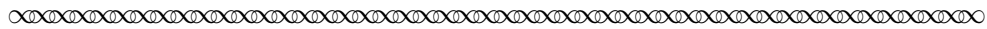
<sup>6</sup>The complete  $16 \times 16$  table can be obtained by assuming the system of octaves satisfies two familiar rules of opposites:  $(-x)y = x(-y) = -(xy)$  and  $(-x)(-y) = xy$  for all  $x, y$ .

	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>
<i>i</i>	<i>i</i>	-1	<i>p</i>	<i>s</i>	- <i>j</i>	<i>r</i>	- <i>q</i>	- <i>k</i>
<i>j</i>	<i>j</i>	- <i>p</i>	-1	<i>q</i>	<i>i</i>	- <i>k</i>	<i>s</i>	- <i>r</i>
<i>k</i>	<i>k</i>	- <i>s</i>	- <i>q</i>	-1	<i>r</i>	<i>j</i>	- <i>p</i>	<i>i</i>
<i>p</i>	<i>p</i>	<i>j</i>	- <i>i</i>	- <i>r</i>	-1	<i>s</i>	<i>k</i>	- <i>q</i>
<i>q</i>	<i>q</i>	- <i>r</i>	<i>k</i>	- <i>j</i>	- <i>s</i>	-1	<i>i</i>	<i>p</i>
<i>r</i>	<i>r</i>	<i>q</i>	- <i>s</i>	<i>p</i>	- <i>k</i>	- <i>i</i>	-1	<i>j</i>
<i>s</i>	<i>s</i>	<i>k</i>	<i>r</i>	- <i>i</i>	<i>q</i>	- <i>p</i>	- <i>j</i>	-1

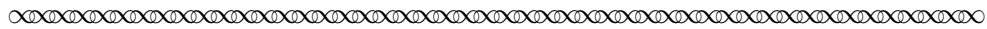
**Task 6** Use the above table to explain why it is impossible to assign a value to the triple product ‘*jkp*’. Indicate how this relates to the lack of associativity for octaves, and comment on why this is inconvenient.

**Task 7** Now find a specific example to show that octave multiplication is also non-commutative.

In describing the properties of the algebraic operation of the *group* structure (see footnote 3), Arthur Cayley wrote the following in his landmark paper [Cayley, 1854]:



But the symbols  $\theta, \phi, \dots$  are in general such that  $\theta.\phi\chi = \theta\phi.\chi$ , &c., so that  $\theta\phi\chi, \theta\phi\chi\omega$ , &c. have a definite signification independent of the particular mode of compounding the symbols; this will be the case even if the functional operations involved in the symbols  $\theta, \phi$ , &c. contain parameters such as the quaternion imaginaries  $i, j, k$ ; but not if these functional operations contain parameters such as the imaginaries which enter into the theory of octaves, and for which, e.g.  $\alpha.\beta\gamma$  is something different from  $\alpha\beta.\gamma$ , a supposition which is altogether excluded from the present paper.



- Task 8**
- (a) In declaring that “the quaternion imaginaries” *can* be part of the elements of a group, what algebraic property was Cayley telling us that a group operation need *not* have?
  - (b) In declaring that “the imaginaries which enter into the theory of octaves” are “altogether excluded from the elements of a group”, what algebraic property was Cayley telling us that a group operation *must* have?

### 3 The Birth of Boolean Algebra in the Laws of Logic

On virtually the same day in 1847, two major new works on logic were published by prominent British mathematicians: Augustus De Morgan (1806–1871) and George Boole (1815–1864). In his work, Boole

sought to stretch the boundaries of traditional syllogistic logic<sup>7</sup> by developing a general method for representing and manipulating *all* logically valid inferences<sup>8</sup> or, as De Morgan described it to Boole in a letter dated 28 November 1847, to develop “mechanical modes of making transitions, with a notation which represents our head work” [Smith, 1982, p. 25]. In contrast to De Morgan, however, Boole took the significant step of explicitly adopting *algebraic* methods for this purpose. As De Morgan himself later proclaimed, “Mr. Boole’s generalization of the forms of logic is by far the boldest and most original . . .” (as quoted in [Merrill, 1990, p. 174]).

In this section of the project, we examine Boole’s bold and original approach to logic and the strange new system of algebra that developed out of it. By reading excerpts from his original writings, we will consider the ways in which the laws of this new algebraic system both resemble and differ from those of standard algebra, as well as the reasons why it satisfies these various laws.

**3.1 Boole’s *Laws of Thought*: The Basic Elements**

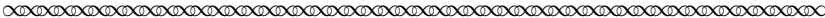
Born in Lincoln England in 1815, Boole was highly influenced by the love of learning that his father, a shoemaker of humble means, passed on to him. Like Hamilton, his early scholarly interest was in languages — he learned Latin, Greek, French, Italian and German largely through self study. At the age of 17, however, he discovered mathematics and began to study works of great mathematicians on his own. Boole’s lack of formal academic credentials prevented him from obtaining a university post for many years. During this period, he conducted mathematical research while working as a school teacher. In 1842, he began a lifelong correspondence with fellow mathematician and logician Augustus De Morgan; their approaches to logic were, however, developed quite independently [Smith, 1982]. In fact, Boole’s earliest mathematical publications lay outside the field of mathematical logic. In 1844, he was awarded a medal by the Royal Society for an article on new techniques in differential equations. He finally obtained a university post in Cork, Ireland in 1849, where he wrote two well-received textbooks on differential equations. It was at Cork that Boole conducted the investigations of logic that led to his mature work on

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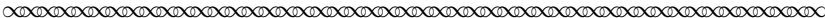
<sup>7</sup>A syllogism is a particular type of logical form consisting of three statements — a major premise, a minor premise (or middle term), and a conclusion based on the two premises — that was first studied by the ancient Greek philosopher Aristotle (384–322 B.C.E.). The following example of a particular syllogistic form is found in the opening paragraph of De Morgan’s *Formal Logic*:

All men will die.	Every Y is X.
All men are rational beings.	Every Y is Z.
Therefore some rational beings will die.	Therefore some Zs are Xs.

<sup>8</sup>Here is how De Morgan described ‘logical validity’ in the opening of his text:

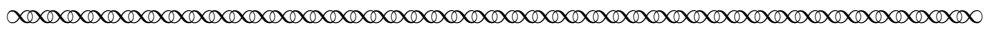


The first notion which a reader can form of Logic is by viewing it as the examination of that part of reasoning which depends upon the manner in which inferences are formed, and the investigation of general maxims and rules for constructing arguments, so that the conclusion may contain no inaccuracy which was not previously asserted in the premises. It has so far nothing to do with the truth of the facts, opinions, or presumptions, from which an inference is derived; but simply takes care that the inference shall certainly be true, if the premises be true. . . . Whether the premises be true or false, is not a question of logic, but of morals, philosophy, history, or any other knowledge to which their subject-matter belongs: the question of logic is, does the conclusion certainly follow if the premises be true? [DeMorgan, 1847, p. 1]



logic, *An Investigation of the Laws of Thought*, published in 1854. Ironically, the book was not initially well received; Boole and a friend who bore the expense of its initial printing probably did not recover their costs. Although he wrote several unpublished manuscripts on logic following the appearance of *Laws of Thought*, and planned another book as its sequel, all of his later publications were on differential equations. At the age of 49, he died of pleuro-pneumonia contracted after walking home 3 miles in a rain storm. His wife Mary Everest Boole (1832–1916), herself a mathematician and teacher, and their five daughters, two of whom later became mathematicians, survived him.

Boole’s work in logic was strongly influenced by the general state of British mathematics at the time, and British Symbolic Algebra in particular.<sup>9</sup> In what follows, we will see the influence of these ideas on Boole’s conception of logic as presented in *Laws of Thought*. We begin with excerpts from Chapter 2 [Boole, 1854, p. 24–38] in which Boole (a) explained the use of ‘signs’ to represent ‘classes,’ (b) defined a system of symbols (+, −, ×, 0, 1) to represent operations on these signs, and (c) deduced the basic laws that these operations must follow.



## AN INVESTIGATION OF THE LAWS OF THOUGHT

### CHAPTER II

1. That Language is an instrument of human reason, and not merely a medium for the expression of thought, is a truth generally admitted. . . .

2. The elements of which all language consists are signs or symbol. Words are signs. . . . But, words . . . are not the only signs which we are capable of employing. Arbitrary marks, which speak only to the eye, and arbitrary sounds or actions, which address themselves to some other sense, are equally of the nature of signs, provided that their representative office is defined and understood. In the mathematical sciences, letters, and the symbols +, −, =, &c., are used as signs . . . . In the present treatise . . . it is with written signs that we have to do, and it is with reference to these exclusively that the term “sign” will be employed. The essential properties of signs are enumerated in the following definition:

*Definition.* – A sign is an arbitrary mark, having a fixed interpretation, and susceptible of combination with other signs in subjection to fixed laws dependent upon their mutual interpretation.

. . .

4. The analysis and classification of those signs by which the operations of reasoning are conducted will be considered in the following Proposition:

#### PROPOSITION I.

*All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.*<sup>10</sup>:

*1st. Literal symbols, as x, y &c., representing things as subjects of our conceptions.*

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<sup>9</sup>While this is not altogether unsurprising, it is worth noting that De Morgan’s approach was not so influenced, a testimony to the genius of Boole’s insight to combine symbolic algebra principles with the study of logic.

<sup>10</sup>“Viz.” is an abbreviation for the Latin word *videlicet*, meaning “namely” or “that is to say”.

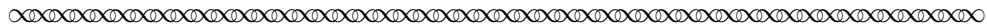
2nd. Signs of operation, as  $+$ ,  $-$ ,  $\times$ , standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the elements.

3rd. The sign of identity,  $=$ .

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

...

6. Now, as it has been defined that a sign is an arbitrary mark, it is permissible to replace all signs of the species above described by letters. Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as  $x$ . ... By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms "nothing" and "universe," which as "classes" should be understood to comprise respectively "no beings," and "all beings." ... Let it further be agreed, that by the combination  $xy$  shall be represented that class of things to which the names or descriptions represented by  $x$  and  $y$  are simultaneously applicable. Thus, if  $x$  alone stands for "white things," and  $y$  for "sheep," let  $xy$  stand for "white sheep;" and in like manner, if  $z$  stand for "horned things," and  $x$  and  $y$  retain their previous interpretations, let  $zxy$  represent "horned white sheep," i.e., that collection of things to which the name "sheep," and the descriptions "white" and "horned" are together applicable. Let us now consider the laws to which the symbols  $x$ ,  $y$ , &c., used in the above sense, are subject.



**Task 9** Boole went on to describe these various "Law of Thoughts" beginning with the following:

First, it is evident, that according to the above combinations, the order in which two symbols are written is indifferent. ... Hence we have,

$$xy = yx. \tag{1}$$

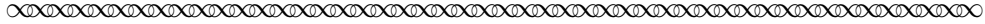
Provide an example of classes  $x$  and  $y$ , different than the example given by Boole in his Section 6 above, to illustrate why this law (i.e., *commutativity*) holds for literal symbols representing classes. Express in English what is meant by  $xy$  and by  $yx$ , and argue that they mean the same thing.

**Task 10** Another standard algebraic law for multiplication is associativity:

$$(xy)z = x(yz) \text{ for all } x, y, z.$$

Provide an example of classes  $x$ ,  $y$  and  $z$  to illustrate why this law also holds for literal symbols representing classes. Then explain how we know that Boole assumed associativity for multiplication in his Section 6 above, even though he did not explicitly mention it.

Boole commented on the analogy between the law of commutativity for symbolic logic and the corresponding law for algebra in his Section 8, which we shall read below. Note his insistence on viewing commutativity as a *formal* (e.g., symbolic) law that happens to hold in two distinct systems (e.g., algebra and logic) in that section. He continued in his Section 9 to then draw on this analogy with algebra in order to deduce another formal law of logic.



8. . . . The law expressed by (1) may be characterized by saying that the literal symbols  $x$ ,  $y$ ,  $z$ , are *commutative, like the symbols of Algebra*. In saying this, it is not affirmed that the process of multiplication in Algebra, of which the fundamental law is expressed by the equation

$$xy = yx,$$

possesses in itself any analogy with that process of logical combination which  $xy$  has been made to represent above; but only that if the arithmetical and the logical process are expressed in the same manner, their symbolical expressions will be subject to the same formal law. The evidence of that subjection is in the two cases is quite distinct.

9. As the combination of two literal symbols in the form  $xy$  expresses the whole of that class of objects to which the names or qualities represented by  $x$  and  $y$  are together applicable, it follows that if the two symbols have exactly the same signification, their combination expresses no more than either of the symbols taken alone would do. In such case we should therefore have

$$xy = x.$$

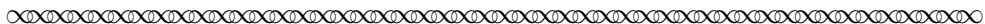
As  $y$  is, however, supposed to have the same meaning as  $x$ , we may replace it in the above equation by  $x$ , and we thus get

$$xx = x.$$

Now in common Algebra the combination  $xx$  is more briefly represented by  $x^2$ . Let us adopt the same principle of notation here; for the mode of expressing a particular succession of mental operations is a thing in itself quite as arbitrary as the mode of expressing a single idea or operation (II. 3). In accordance with this notation, then, the above equation assumes the form

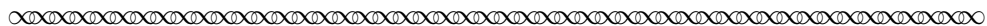
$$x^2 = x, \tag{2}$$

and is, in fact, the expression of a second general law of those symbols by which names, qualities, or descriptions, are symbolically represented.



**Task 11** Use a specific example of a class  $x$  to explain why the symbolic law “ $x^2 = x$ ” makes sense as a property of classes under Boole’s definition of multiplication. (Remember that  $xy$  represents the class of things that simultaneously possess the quality  $x$  and the quality  $y$ .) What about the symbolic law “ $x^3 = x$ ”? Do either of the laws “ $x^2 = x$ ” or “ $x^3 = x$ ” also hold in arithmetical algebra? Explain.

We now move to Boole's discussion of how the operations '+' and '-' apply in his *Laws of Thought*. Notice how his introduction "-" again used analogy with algebra as motivation. Following a discussion of the equality symbol "=" to represent the verb *is* or *are* in his Section 12 (omitted here), Boole then further explored the analogy of symbolic logic and algebra in Section 13 below.



11. *Signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts.*

We are not only capable of entertaining the conceptions of objects, as characterized by names, qualities, or circumstances, applicable to each individual of the group under consideration, but also of forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described. For this purpose we use the conjunctions "and," "or," &c. "Trees and minerals," "barren mountains, or fertile vales," are examples of this kind. In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws identical. Thus the expression "men and women" is, conventional meanings set aside, equivalent with the expression "women and men." Let  $x$  represent "men,"  $y$ , "women;" and let + stand for "and" and "or," then we have

$$x + y = y + x, \tag{3}$$

an equation which would equally hold true if  $x$  and  $y$  represented numbers, and + were the sign of arithmetical addition.

...

... The above are the laws which govern the use of the sign +, here used to denote the positive operation of aggregating parts into a whole. But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done. Thus we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from the whole. This operation we express in common language by the sign *except*, as, "All men except Asiatics," "All states except those which are monarchical." Here it is implied that the things excepted form a part of the things from which they are excepted. As we have expressed the operation of aggregation by the sign +, so we may express the negative operation above described by - minus. Thus if  $x$  be taken to represent men, and  $y$ , Asiatics, i.e., Asiatic men, then the conception of "All men except Asiatics" will be expressed by  $x - y$ . And if we represent by  $x$ , "states," and by  $y$  the descriptive property "having a monarchical form," then the conception of "All states except those which are monarchical" will be expressed by  $x - xy$ .

...

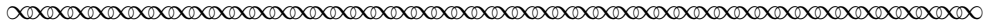
13. Let us take the Proposition, "The stars are the suns and the planets," and let us represent stars by  $x$ , suns by  $y$ , and planets by  $z$ ; we have then

$$x = y + z. \tag{7}$$

Now if it be true that the stars are the suns and the planets, it will follow that the stars, except the planets, are suns. This would give the equation

$$x - z = y, \tag{8}$$

which must therefore be a deduction from (7). Thus a term  $z$  has been removed from one side of an equation to the other by changing its sign. This is in accordance with the algebraic rule of transposition.



**Task 12** This task examines the *two* distributivity rules that hold in Boole’s system.

- (a) Elsewhere in his Section 11, Boole justified the distributivity law for multiplication over addition,

$$z(x + y) = zx + zy, \tag{4}$$

by letting  $x$  represent “men,”  $y$  represent “women,” and  $z$  represent “European.” Describe, in English, the meaning of each side of the equation for this particular example. Is this a convincing justification for the law? If not, how else might we justify it?

- (b) Now take a look at the less familiar distributivity law for multiplication over addition,

$$x + (yz) = (x + y)(x + z).$$

Again letting  $x$  represent “men,”  $y$  represent “women,” and  $z$  represent “European,” describe, in English, the meaning of each side of the equation for this particular example to provide at least a partial justification for this law. Provide additional examples or some other type of justification, as needed, to convince yourself that it holds.

Note that Boole imposed a restriction on the use of the addition symbol “+” in his system by asserting that “the words ‘and,’ ‘or,’ interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another.” Today, we refer to classes with no common members as *disjoint classes* (or *disjoint sets*).<sup>11</sup> Despite Boole’s insistence on restricting the operation + (or union) to only disjoint classes, this restriction is no longer imposed on a Boolean algebra today.<sup>12</sup> Similarly, Boole’s restriction on the operation ‘−’ which required that “the things excepted form a part of the things from which they are excepted,” while sensible from the point of view of common language usage, is no longer a consideration in today’s abstract Boolean algebra. In

<sup>11</sup> In a now-standard diagram named after John Venn (1834–1923), two disjoint sets are represented by two non-overlapping circles. A Venn diagram for sets  $x, y$  that are not disjoint consists of two overlapping circles, with their overlapping region representing the intersection  $xy$  of the sets. The region covered by both sets together (including their overlap) represents the union  $x + y$  of the sets.

<sup>12</sup>In Section 11 of his *Laws of Thought*, Boole referred to the analogy of symbolic logic with arithmetical algebra as further justification for restricting the use of “+” to disjoint classes:

In strictness, the words “and,” “or,” interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words “and” “or” are analogous with the sign + in algebra, and their laws identical.

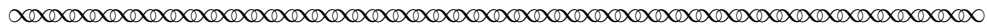
In other words, for the definition of addition for whole numbers  $m, n$  the sum  $m + n$  to make sense, it is important that  $m + n$  be defined to be the total number of elements in the union (or aggregate) of two *disjoint* sets containing  $m$  objects with a set containing  $n$  objects. You should be able to easily find specific examples to illustrate why it is important to use disjoint sets in this definition. In fact, even Boole had a way to work around this restriction when he needed to do so in his *Laws of Thought*. It was Boole’s follower John Venn (see footnote 11) who was one of the first to fully drop the restriction in his own study of Boole’s work in the late nineteenth century. See Section 3 of the student project [Barnett, 2013c] for more details.

what follows, you thus need not be overly concerned about checking that these restrictions hold before “adding” or “subtracting” two classes. In fact, Task 13 begins to explore how we could (and Boole did) work around the stated restriction on ‘−’ within his system.

**Task 13** In this task, we consider Boole’s comment (in his Section 11) that the operation *minus* will require “that the things excepted form a part of the things from which they are excepted.”

- (a) For example, if  $x$  represents men and  $y$  represents women, then the expression  $x - y$  would have been meaningless in Boole’s system since the class of women does not form part of the class of men. Describe the class that the expression  $x - y$  would denote if we ignore Boole’s restriction.
- (b) Similarly, in the case where  $d$  represents drummers and  $p$  represents pianists, Boole would have considered the expression  $d - p$  to be meaningless since the class of drummers does not include all of the class of pianists. Describe the class that the expression  $d - p$  would denote if we ignore Boole’s restriction. Based on your reading of Boole so far, how could we legitimately represent this class within his system? Explain.

We now pick up our reading of Boole’s Chapter II where we left off, meeting yet another law of Boole’s system that differs from that of arithmetical algebra.



13. ... Again: If two classes of things,  $x$  and  $y$ , be identical, that is, if all members of the one are members of the other, then those members of the one class which possess a given property  $z$  will be identical with those members of the other which possess the same property  $z$ . Hence if we have the equation

$$x = y,$$

then whatever class or property  $z$  may represent, we have also

$$zx = zy.$$

This is formally the same as the algebraic law:– If both members of an equation are multiplied by the same quantity, the products are equal.

In like manner it may be shown that if the corresponding members of two equations are multiplied together, the resulting equation is true.

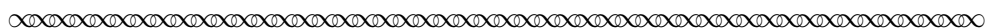
14. Here, however, the analogy of the present system with that of algebra, as commonly stated, appears to stop. Suppose it true that those members of a class  $x$  which possess a certain property  $z$  are identical with those members of a class  $y$  which possess the same property  $z$ , it does not follow that the members of the class  $x$  universally are identical with the members of the class  $y$ . Hence it cannot be inferred from the equation

$$zx = zy$$

that the equation

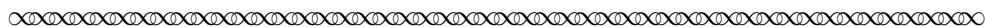
$$x = y$$

is also true. In other words, the axiom of algebraists, that both sides of an equation may be divided by the same quantity, has no formal equivalent here.



**Task 14** Provide a specific example of classes  $x$ ,  $y$  and  $z$  to illustrate that Boole’s assertion in his Section 14 above is correct; that is, show that it is possible to have  $zx = zy$  for  $x \neq y$  and  $z \neq 0$ .

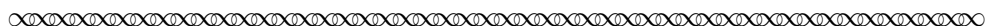
In the closing Section (15) of Chapter II of *Laws of Thought*, Boole identified a special ‘algebra of number’ with properties that are strongly analogous to those exhibited by operations on classes. In Chapter III, Boole then derived several additional laws for symbolic logic from the analogy with this specific “arithmetical algebra.” In our reading from Chapter III [Boole, 1854, pp. 47–51], we focus on three specific propositions in which he provided interpretations for the symbols “0”, “1”, and “ $1 - x$ ”, and presented the first formal deduction of a logical principle in mathematical form.



15. ... We have seen that the symbols of Logic are subject to the special law,

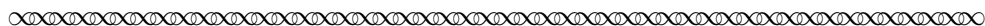
$$x^2 = x.$$

Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that  $0^2 = 0$ , and that  $1^2 = 1$ ; and the equation  $x^2 = x$ , considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity admitting only of the values 0 and 1. Let us conceive, then, of an Algebra in which the symbols  $x$ ,  $y$ ,  $z$ , &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.



**Task 15** Recall that for classes, the expression ‘ $1 + 1$ ’ was meaningless for Boole. (*Do you remember why?*) Ignoring Boole’s restriction on the use of ‘+’, which of the two values 0 or 1 would it make sense to assign to the expression ‘ $1 + 1$ ’ as a statement about classes? Explain. If we were to instead consider the expression ‘ $1 + 1$ ’ as a *numerical* statement within the ‘Algebra of Number,’ does it make sense to assign either 0 or 1 to the sum  $1 + 1$ ? Why or why not?

Returning again to our reading of Boole, we continue to his Chapter III.



### CHAPTER III

#### PROPOSITION II

*To determine the logical value and significance of the symbols 0 and 1.*  
The symbol 0, as used in Algebra, satisfies the following formal law,

$$0 \times y = 0, \text{ or } 0y = 0, \tag{1}$$

whatever number  $y$  may represent. That this formal law may be obeyed in the system of Logic, we must assign to the symbol 0 such an interpretation that the class represented by  $0y$  may be identical with the class represented by 0, whatever the class  $y$  may be. A little consideration will show that this condition is satisfied if the symbol 0 represent Nothing. . . .

Secondly, The symbol 1 satisfies in the system of Number the following law, viz.,

$$1 \times y = y, \text{ or } 1y = y,$$

whatever number  $y$  may represent. And this formal equation being assumed as equally valid in the system of this work, in which 1 and  $y$  represent classes, it appears that the symbol 1 must represent such a class that all the individuals which are found in any proposed class  $y$  are also all the individuals  $1y$  that are common to that class  $y$  and the class represented by 1. A little consideration will here show that the class represented by 1 must be "the Universe," since this is the only class in which are found all the individuals that exist in any class. Hence the respective interpretations of the symbols 0 and 1 in the system of Logic are Nothing and Universe.

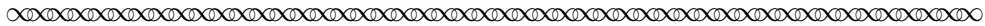
. . .

### PROPOSITION III

*If  $x$  represent any class of objects, then will  $1 - x$  represent the contrary or supplementary class of objects, i. e., the class including all objects which are not comprehended in the class  $x$ .*

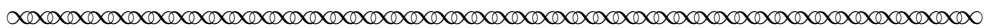
For greater distinctness of conception let  $x$  represent the class men, and let us express, according to the last Proposition, the Universe by 1; now if from the conception of the Universe, as consisting of "men" and "not-men," we exclude the conception of "men," the resulting conception is that of the contrary class, "not-men." Hence the class "not-men" will be represented by  $1 - x$ . And, in general, whatever class of objects is represented by the symbol  $x$ , the contrary class will be expressed by  $1 - x$ .

. . .



**Task 16** Recall the classes used in Task 13 above:  $d$  represents drummers, and  $p$  represents pianists. Again ignoring Boole's restriction that subtraction is only meaningful if "the things excepted form a part of the things from which they are excepted," use the notation introduced in Propositions II and III above to symbolically represent the class denoted by the expression  $d - p$ . Explain briefly.

Boole continued:



### PROPOSITION IV

*That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is  $x^2 = x$ .*

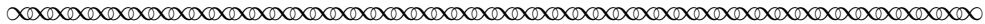
Let us write this equation in the form

$$x - x^2 = 0,$$

whence we have

$$x(1 - x) = 0; \tag{1}$$

both these transformations being justified by the axiomatic laws of combination and transposition (II. 13). Let us, for simplicity of conception, give to the symbol  $x$  the particular interpretation of men, then  $1 - x$  will represent the class of "not-men" (Prop. III.) Now the formal product of the expressions of two classes represents that class of individuals which is common to them both (II. 6). Hence  $x(1 - x)$  will represent the class whose members are at once "men" and "not-men," and the equation (1) thus express the principle, *that a class whose members are at the same time men and not men does not exist*. In other words, that *it is impossible for the same individual to be at the same time a man and not a man*. Now let the meaning of the symbol  $x$  be extended from the representing of "men," to that of any class of beings characterized by the possession of any quality whatever; and the equation (1) will then express that it is impossible for a being to possess a quality and not to possess that quality at the same time. But this is identically that "principle of contradiction" which Aristotle has described as the fundamental axiom of all philosophy.



**Task 17** In outline form, Boole's proof of Proposition IV is as follows:

$$\begin{aligned} \text{Given:} & \quad x = x^2. \\ \text{Then} & \quad x - x^2 = 0, \\ & \quad \text{and} \quad x(1 - x) = 0. \end{aligned}$$

Interpreting the equation  $x(1 - x) = 0$ , we conclude that the classes  $x$  and  $1 - x$  have no common members. Thus, it is impossible for "any being to possess a quality, and at the same time not to possess it."

In his discussion of Proposition IV, Boole made the following additional comments (emphasis added):

The above interpretation has been introduced not on account of its immediate value in the present system, but *as an illustration of a significant fact* in the philosophy of the intellectual powers, viz., *that what has been commonly regarded as the fundamental axiom of metaphysics is but the consequence of a law of thought, mathematical in its form.*

Does Boole's formal manipulation of the law  $x^2 = x$  provide convincing proof that the *Principle of Contradiction* is a 'consequence of a law of thought, mathematical in its form,' rather than being an 'axiomatic law'? Explain.

**Task 18** In a footnote to Proposition IV (omitted here), Boole discussed the status of the equation  $x^3 = x$  in his system, and remarked that this equation can be factored as follows:

$$\begin{aligned}x(1 - x)(1 + x) &= 0 \\x(1 - x)(-1 - x) &= 0\end{aligned}$$

He then concluded that  $x^3 = x$  is ‘not interpretable in the system of logic’ because both factorizations contain meaningless factors,  $1 + x$  and  $-1 - x$  respectively. Comment on why these two expressions ( $1 + x$  and  $-1 - x$ ) were meaningless in Boole’s system. [*Hints?* Consider (a) Boole’s definition of addition, and (b) the symbol  $-1$  relative to the second degree law  $x^2 = x$ .] Then provide an argument *for* considering the equation  $x^3 = x$  to be a valid law of symbolic logic.

While there is considerably more that we could read from Boole’s work, we now turn towards a much later work that shows how his creation was eventually put to work in the problem of circuit design.

## 4 Claude Shannon, Boolean Algebra and Circuit Design

As we have seen, Boole’s bold and original approach to logic was to use a symbolic algebra with two symbols ( $\times, +$ ) representing operations on classes, or sets. He then interpreted these symbols in a way that corresponds to connectives in today’s propositional logic, with multiplication ( $xy$ ) corresponding to conjunction (‘and’) and addition ( $x + y$ ) to disjunction (‘or’). Under these interpretations, certain algebraic laws must hold. Many of these laws also hold in standard algebra; for example,  $x + y = y + x$  and  $x(y + z) = xy + xz$ . But other rules of of this algebra — such as  $x + yz = (x + y)(x + z)$  — are quite strange looking! Today, any system that satisfies the complete set of laws set forth in Boole’s ‘Algebra of Logic’ is called a *Boolean algebra*.

One of Boolean algebra’s strange new laws is called the *Idempotent Law*:  $x^2 = x$ . Of course, the Idempotent Law *does* hold in standard algebra, but only for the two values  $x = 0$  and  $x = 1$ . In the early twentieth century, this special ‘two-valued algebra’ was an important part of mathematicians’ efforts to find axioms on which to base the study of abstract Boolean algebras. The American mathematician Edward V. Huntington, for example, used the following tables to define addition and multiplication (which he denoted by  $\oplus$  and  $\odot$  respectively) as part of a completely abstract (i.e., meaningless) model for such an axiom system [Huntington, 1904, p. 293]:

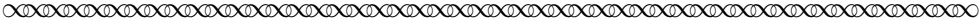
$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \odot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

In this section, we will see how this same two-valued system came to be used in a very concrete application of Boolean algebra: the design and analysis of circuits.

The individual who applied Boolean algebra to study circuit design was American mathematician and electrical engineer Claude E. Shannon (1916–2001). Shannon completed bachelor degrees in both mathematics and electrical engineering at the University of Michigan in 1936. Two years later, at the age of 22, he completed a master’s thesis in electrical engineering at the Massachusetts Institute of Technology. The idea that inspired his thesis work came from his exposure to symbolic logic in an undergraduate philosophy course. Vannever Bush (1890–1974), dean of engineering at MIT and inventor

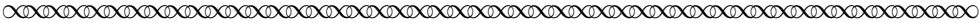
an early mechanical computer called the differential analyser machine, was so impressed by Shannon's thesis that he sponsored its publication in an engineering journal in 1938. This award-winning paper, *A Symbolic Analysis of Relay and Switching Circuits*, revolutionized the study of switches and relays, which in turn form the circuitry behind the binary arithmetic of modern computers.<sup>13</sup>

Shannon described the problem and his proposed solution as follows [Shannon, 1938, p. 713]:

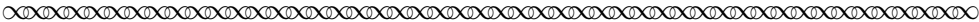


In the control and protective circuits of complex electrical systems it is frequently necessary to make intricate interconnections of relay contacts and switches. Examples of these circuits occur in automatic telephone exchanges, industrial motor-control equipment, and in almost any circuits designed to perform complex operations automatically. In this paper a mathematical analysis of certain of the properties of such networks will be made. . . .

The method of attack on these problems may be described briefly as follows: any circuit is represented by a set of equations, the terms of the equations corresponding to the various relays and switches in the circuit. A calculus is developed for manipulating these equations by simple mathematical processes, most of which are similar to ordinary algebraic algorithms. This calculus is shown to be exactly analogous to the calculus of propositions used in the symbolic study of logic.

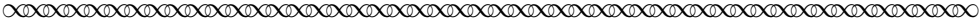


The key to applying symbolic Boolean algebra to relay and switching circuits lay in the fact that there are only two possible states for such circuits, open and closed, a situation reminiscent of Huntington's two-valued Boolean algebra model, 0 and 1. In fact, Shannon began his paper by stating the following arithmetical postulates for circuits — look back at the tables given by Huntington to be sure you see how Shannon's postulates match the operations in those tables.



**Postulates**

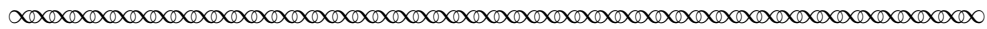
- 1. a.  $1 + 1 = 1$   
b.  $0 \cdot 0 = 0$
- 2. a.  $1 + 0 = 0 + 1 = 1$   
b.  $0 \cdot 1 = 1 \cdot 0 = 0$
- 3. a.  $0 + 0 = 0$   
b.  $1 \cdot 1 = 1$
- 4. At any given time either  $X = 0$  or  $X = 1$ .



---

<sup>13</sup>Shannon went on to complete a doctorate in mathematics at MIT with a thesis on the application of mathematics to genetics, and began his official career as a research mathematician at Bell Laboratories in 1941. His association with Bell Labs continued until 1971. In 1948, he published yet another ground breaking paper, *A Mathematical Theory of Communication*, thereby launching the still flourishing field of information theory. He married in 1949 (he and his wife had four children), and served as a faculty member and researcher at MIT from 1956 through 1978. Shannon is also remembered for various mechanical inventions, and for his successful stock investment strategies. Among his many honors was the first ever Marconi Lifetime Achievement Award, awarded to him in 2000. By this time, sadly, Shannon suffered significantly from the effects of Alzheimer's disease; he died in a nursing home just a year later.

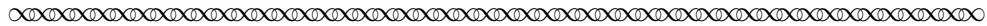
In a 1987 interview with *Omni* magazine [Sloane and Wyner, 1993], Shannon was asked the question ‘Was the basic insight that yes/no can be embodied in on/off switches so trivial?’ His response described the analogy between circuits and Boolean algebra as follows:



It’s not so much that a thing is “open” or “closed,” the “yes” or “no” that you mentioned. The real point is that two things in series are described by the word “and” in logic, so you would say this “and” this, while two things in parallel are described by the word “or.”

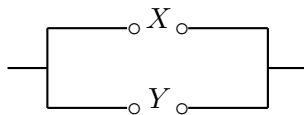
The people who had worked with relay circuits were, of course, aware of how to make these things. But they didn’t have the mathematical apparatus of the Boolean algebra to work with them, and to do them efficiently. . . .

They all knew the simple fact that if you had two contacts in series both had to be closed to make a connection through. Or if they are in parallel, if either one is closed the connection is made. They knew it in that sense, but they didn’t write down equations with plus and times, where plus is like a parallel connection and times is like a series connection.

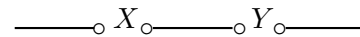


**Task 19** The diagrams below show how Shannon used circuits to physically represent symbolic operations on 0 and 1, where “plus is like a parallel connection and times is like a series connection.”

Parallel Connection:  $X + Y$



Series Connection:  $X \cdot Y$



- (a) For a parallel connection, Shannon stated “the connection is made if either contact  $X$  or  $Y$  is closed.” By his postulates, this must give three addition facts:

$$0 + 0 = 0 \quad ; \quad 0 + 1 = 1 + 0 = 1 \quad ; \quad 1 + 1 = 1$$

Which number (‘0’ or ‘1’) must mean that a contact is closed (i.e., connection is made)? Explain.

- (b) For the series connection, both contacts  $X$  and  $Y$  must be closed in order to make a connection through the overall circuit. Explain how this relates to Shannon’s multiplication postulates by explaining the following:

Why is  $0 \cdot 0 = 0$ ?

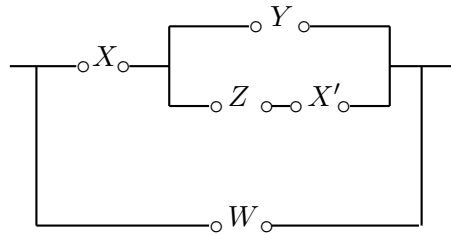
Why is  $0 \cdot 1 = 1 \cdot 0 = 0$ ?

Why is  $1 \cdot 1 = 1$ ?

#### 4.1 Expressing Network Diagrams Algebraically

An example of how to represent a more complicated circuit with an algebraic expression, based on an example from Shannon [Shannon, 1949, p. 589], is shown in the following figure.

Here, Shannon used the notation  $X'$  to represent the ‘negative of  $X$ ’, or ‘not- $X$ ’; thus, the contact  $X'$  is closed (connection made between terminal points) whenever the contact  $X$  is open (no connection made between terminal points), and vice versa.

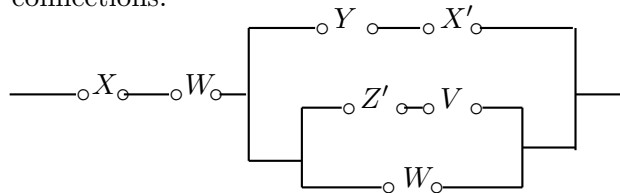


Network for Boolean expression  $W + X \cdot (Y + Z \cdot X')$

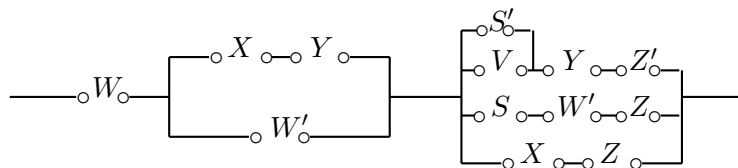
**Task 20** Consider the network shown above.

- Explain why the overall network is closed when contact  $W$  is closed (i.e., connection is made), regardless of the states of contacts  $X$ ,  $Y$  and  $Z$ .
- Now determine whether this network is open or closed when contact  $W$  is open and contacts  $X$ ,  $Y$  and  $Z$  are closed; explain your conclusion.

**Task 21** Represent the network shown below by a Boolean expression, using  $+$  for parallel connections and  $\cdot$  series for connections.



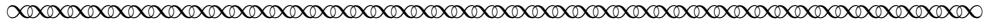
**Task 22** Represent the network shown below (adapted from [Shannon, 1938, p. 715]) by a Boolean expression, using  $+$  for parallel connections and  $\cdot$  series for connections.



**Task 23** Sketch the network represented by the Boolean expression  $X + Y(Z + W) + X'Z$ , again using  $+$  for parallel connections and  $\cdot$  series for connections.

## 4.2 Using Networks to Prove Boolean Algebra Identities

Shannon used his diagrams not only to represent given circuits, but also to illustrate Boolean algebra identities. The following excerpt gives his description of the basic Boolean identities for circuits [Shannon, 1938, p. 713–714].



$$X + Y = Y + X \quad (1a)$$

$$XY = YX \quad (1b)$$

$$X + (Y + Z) = (X + Y) + Z \quad (2a)$$

$$X(YZ) = (XY)Z \quad (2b)$$

$$X(Y + Z) = XY + XZ \quad (3a)$$

$$X + YZ = (X + Y)(X + Z) \quad (3b)$$

$$1 \cdot X = X \quad (4a)$$

$$0 + X = X \quad (4b)$$

$$1 + X = 1 \quad (5a)$$

$$0 \cdot X = 0 \quad (5b)$$

...

Due to the associative laws (2a and 2b) parentheses may be omitted in a sum or product of several terms without ambiguity. ...

The distributive law (3a) makes it possible to “multiply out” products and to factor sums. The dual of this theorem (3b), however, is not true in numerical algebra.

... The negative of a hindrance  $X$  will be written  $X'$  and is defined as a variable which is equal to 1 when  $X$  equals 0 and equal to 0 when  $X$  equals 1. ... The definition of the negative of a hindrance gives the following theorems:

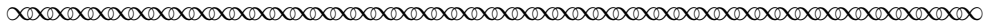
$$X + X' = 1 \quad (6a)$$

$$XX' = 0 \quad (6b)$$

$$0' = 1 \quad (7a)$$

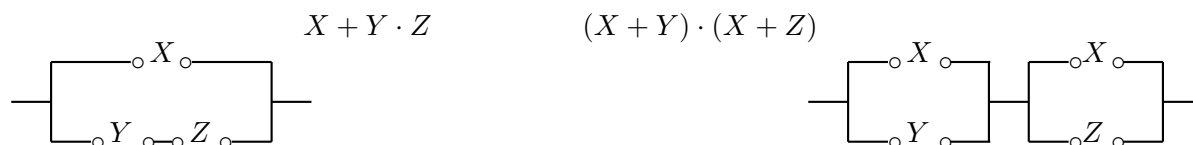
$$1' = 0 \quad (7b)$$

$$(X')' = X \quad (8)$$



Many of the laws listed above by Shannon are familiar from standard algebra. The most unfamiliar law, perhaps, is his (3b):  $X + YZ = (X + Y)(X + Z)$ . The fact that (logical) addition is distributive over (logical) multiplication was, however, a familiar property of Boolean algebra for Shannon and his mathematical predecessors. The next task examines this law from the perspective of circuits.

**Task 24** The figure below (adapted from [Shannon, 1949, p. 591]) uses network diagrams to illustrate the distributivity law for addition over multiplication (3b).



(a) Explain why these two circuits are equivalent by determining what configurations of open (no connection made) and closed contact (connection is made) contacts are needed for flow across (connection made) the overall network.

Flow across $X + Y \cdot Z$	Flow across $(X + Y) \cdot (X + Z)$

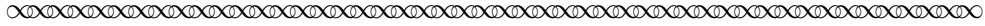
(b) Now complete the following table<sup>14</sup> to show that the two Boolean expressions are equivalent for all possible values of the variables in the Boolean algebra on  $\{0, 1\}$ .

$X$	$Y$	$Z$	$YZ$	$X + Y$	$X + Z$	$X + YZ$	$(X + Y)(X + Z)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

(c) Parts a and b give two different methods for proving two networks are equivalent. Which method of proof do you prefer, and why?

<sup>14</sup>Shannon called the technique of proving two networks are equivalent by directly verifying possible through the use of a table the ‘method of perfect induction’ [Shannon, 1938, p. 714].

Here are some other important properties of a Boolean algebra that Shannon used<sup>15</sup> to design and simplify networks in his paper [Shannon, 1938, pp. 714–715]:



$$(X + Y + Z + \dots)' = X' \cdot Y' \cdot Z' \dots \quad \text{(9a)}$$

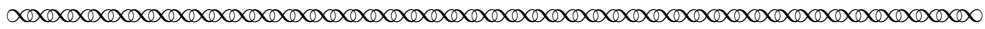
$$(X \cdot Y \cdot Z \dots)' = X' + Y' + Z' \dots \quad \text{(9b)}$$

$$X = X + X = X + X + X = \text{etc.} \quad \text{(14a)}$$

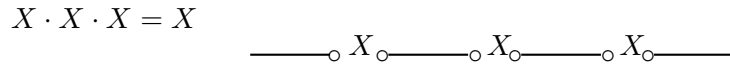
$$X = X \cdot X = X \cdot X \cdot X = \text{etc.} \quad \text{(14b)}$$

$$X + XY = X \quad \text{(15a)}$$

$$X(X + Y) = X \quad \text{(15b)}$$



Notice that Property 14b is the general Idempotent Law for multiplication that we first met in our reading of Boole. In terms of networks, it says that a series of  $X$  contacts of any length can simply be replaced by just a single  $X$  contact. Consider the following diagram to convince yourself this is true.



**Task 25** This task explores the Laws of Absorption (15ab).

- (a) Sketch the network for the left-hand side of the first Absorption Law:  $X + XY = X$ . Explain why this means that network can just be replaced by a single  $X$  contact. Then complete the following table to show the validity of this property within the context of circuits.<sup>16</sup>

$X$	$Y$	$XY$	$X + XY$
0	0		
0	1		
1	0		
1	1		

Use 1 to 3 sentences to explain why this proves the property is valid.

<sup>15</sup>Shannon proved that these laws would hold for circuits by showing that the two-valued algebra of circuits did satisfied all the axioms of a general Boolean algebra. As a consequence, any property that could be proven for a general Boolean algebra must necessarily also hold for the specific two-valued algebra of circuits.

<sup>16</sup>As Shannon noted, the “Perfect Induction” proof technique illustrated by the use of tables (see note 3) is helpful in the context of circuits precisely because ‘each variable is limited to just two values’ [Shannon, 1938, p. 714]. Since this is not the case for all Boolean algebras, establishing these identities in general required more sophisticated proof techniques, such as those used in Huntington’s 1904 paper on the axiomatization of Boolean algebra.

(b) Complete the following table to prove the second Absorption Law:  $X(X + Y) = X$ .

$X$	$Y$	$X + Y$	$X(X + Y)$
0	0		
0	1		
1	0		
1	1		

Compare the tables in parts a and b. What do you notice?

(c) Here is another (algebraic) proof for the second Absorption Law:

$$\begin{array}{lcl}
 X(X + Y) & = & XX + XY & \text{By Distributive Law 3a} \\
 & = & X + XY & \text{By Idempotent Law 9a}
 \end{array}$$

Use this format to give another proof of the Distributivity Law (3b):

$$(X + Y)(X + Z) = X + YZ.$$

$$\begin{array}{lcl}
 (X + Y)(X + Z) & = & (X + Y)X + (X + Y)Z & \text{By Distributive Law 3a} \\
 & = & X(X + Y) + Z(X + Y) & \text{By } \underline{\hspace{4cm}} \\
 & = & &
 \end{array}$$

- (d) Parts a and b give two different methods for proving certain algebraic laws hold within the context in which Shannon was working. Which method of proof do you prefer, and why?
- (e) How do the proof methods that you looked at in part (c) compare to Boole's approach to proving these same algebraic laws? Do you prefer Boole's approach to either of the other two method, and why?

**Task 26** This task explores De Morgan's Laws (9ab).

For two variables  $X$  and  $Y$ , the first of De Morgan's Law is the equality:  $(X + Y)' = X' \cdot Y'$ . Complete the following table to show that this law holds for circuits.

$X$	$Y$	$X + Y$	$(X + Y)'$	$X'$	$Y'$	$X' \cdot Y'$
0	0					
0	1					
1	0					
1	1					

Now complete a table for proving the second of De Morgan's Law:  $(X \cdot Y)' = X' + Y'$ .

$X$	$Y$					
0	0					
0	1					
1	0					
1	1					

**Task 27** In his paper [Shannon, 1938, p. 715], Shannon remarked that De Morgan's Law can be applied to any number of sets. For instance,  $(X + Y + W)' = X'Y'W'$  and  $(X + Y + W + Z)' = X'Y'W'Z'$ .

He also described this generalization as a (mechanical) method by which

the negative of any function may be obtained by replacing each variable by its negative and interchanging the  $+$  and  $\cdot$  rules. ... For example, the negative of  $X + Y \cdot (Z + WX')$  will be  $X'[Y' + Z'(W' + X)]$ .

Without Shannon's method, negation of complicated expressions requires iterative applications of the basic two-set versions of De Morgan's Laws, as shown in the following example:

$$\begin{aligned}
 [X + Y \cdot (Z + WX')] &= X' \cdot \underbrace{[Y \cdot (Z + WX')]'} \\
 &= X' \cdot \left[ Y' + \underbrace{(Z + WX')}' \right] \\
 &= X' \cdot \left[ Y' + (Z' \cdot \underbrace{(WX')}' \right] \\
 &= X' \cdot [Y' + Z' \cdot (W' + X'')] \\
 &= X' \cdot [Y' + Z' \cdot (W' + X)]
 \end{aligned}$$

Re-write each of the following by first using Shannon's method, and then via the iterative De Morgan's Law technique.

(a)  $[XY' + Z]'$

- \* Using Shannon's method;
- \* Using the iterative De Morgan's Law technique.

(b)  $[XW(Y' + Z)]'$

- \* Using Shannon's method;
- \* Using the iterative De Morgan's Law technique.

(c) Which of these two methods do you prefer, and why?

Use your preferred method, and other Boolean algebra properties, to prove the following:

$$[X'(YZ)]' + [YW(X' + Z)]' = X + Y' + Z'.$$

### 4.3 Duality

Notice that Shannon arranged most Boolean algebra properties in pairs:

Postulates	Theorems
(1a) $0 \cdot 0 = 0$	$X + Y = Y + X$ (1a)
(1b) $1 + 1 = 1$	$XY = YX$ (1b)
(2a) $1 + 0 = 0 + 1 = 1$	$X + (Y + Z) = (X + Y) + Z$ (2a)
(2b) $0 \cdot 1 = 1 \cdot 0 = 0$	$X(YZ) = (XY)Z$ (2b)
(3a) $0 + 0 = 0$	$X(Y + Z) = XY + XZ$ (3a)
(3b) $1 \cdot 1 = 1$	$X + YZ = (X + Y)(X + Z)$ (3b)
	$1 \cdot X = X$ (4a)
	$0 + X = X$ (4b)
	$1 + X = 1$ (5a)
	$0 \cdot X = 0$ (5b)
	$X + X' = 1$ (6a)
	$XX' = 0$ (6b)
	$0' = 1$ (7a)
	$1' = 0$ (7b)
	$(X + Y + Z + \dots)' = X' \cdot Y' \cdot Z' \dots$ (9a)
	$(X \cdot Y \cdot Z \dots)' = X' + Y' + Z' \dots$ (9b)
	$X = X + X = X + X + X = \text{etc.}$ (14a)
	$X = X \cdot X = X \cdot X \cdot X = \text{etc.}$ (14b)
	$X + XY = X$ (15a)
	$X(X + Y) = X$ (15b)

A bit later in his paper, Shannon explained that he did this “to emphasize a duality relationship between the operations of addition and multiplication and the quantities zero and one.” In this section, we use circuits to explore this concept of ‘duality’ in a Boolean algebra.

**Task 28** Carefully examine Shannon's dual pairs of postulates and theorems.

- (a) Write a short description of how to obtain the dual of a statement.  
 (b) For each of the expressions given below, illustrate your method by writing the dual of the given expressions. Also represent the given expression and its dual by networks.

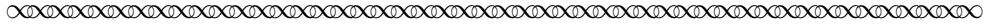
	Given	Dual
Expression	$X + YZ$	
Network		

	Given	Dual
Expression	$(X + Y)(Z + W)$	
Network		

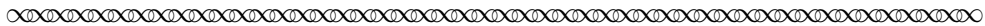
	Given	Dual
Expression	$XZ' + Y + W'$	
Network		

## 4.4 Using Boolean Algebra to Simplify Circuits

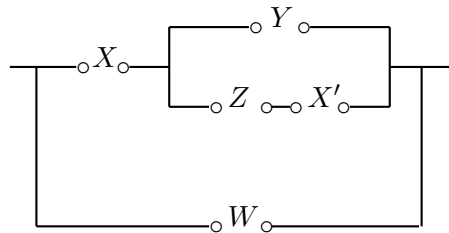
One of Shannon’s goals in applying Boolean algebra to the study of circuits was to use algebraic techniques to simplify complicated systems, as he described in the following excerpt from his 1949 paper “The Synthesis of Two-Terminal Switching Circuits” [Shannon, 1949, p. 590].<sup>17</sup>



By means of Boolean Algebra it is possible to find many circuits equivalent in operating characteristics to a given circuit. The hindrance of the given circuit is written down and manipulated according to the rules. Each different resulting expression represents a new circuit equivalent to the given one. In particular, expressions may be manipulated to eliminate elements which are unnecessary, resulting in simple circuits.



Consider, for example, the following network:



$$W + X \cdot (Y + Z \cdot X')$$

If you look carefully at the network diagram, you might notice that  $Z$  can be eliminated completely.

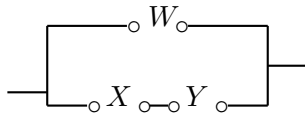
Shannon’s idea for how to show this was to compute as follows:

$$\begin{aligned} W + X \cdot (Y + Z \cdot X') &= W + XY + X(ZX') \\ &= W + XY + (XX')Z \\ &= W + XY + 0 \cdot Z \\ &= W + XY \end{aligned}$$

Not only does this computation show that  $Z$  can be omitted — it also gives us a much simpler network that we can use to replace the original complicated network:

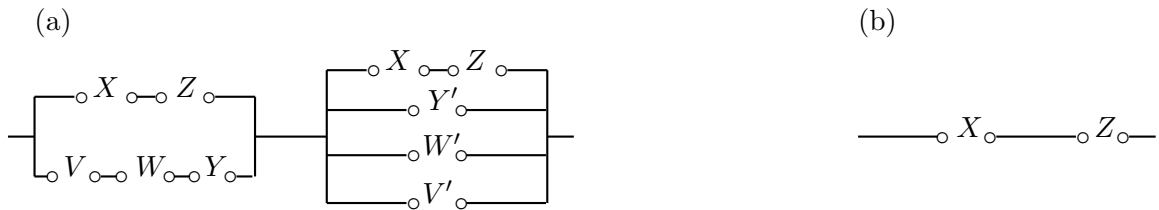
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<sup>17</sup>Another of Shannon’s goals, as suggested by the title of his 1949 paper, was to solve the problem of “synthesis” for circuits: given a specific set of desired inputs and outputs, construct a network of series and parallel connections corresponding to those values. Although that portion of Shannon’s work goes beyond the scope of the current project, you can learn more about it in the student project [Barnett, 2013a].



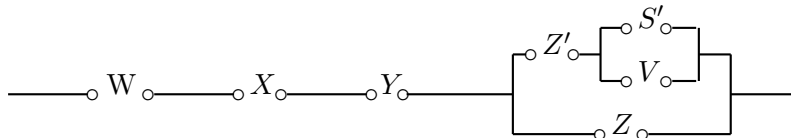
A network equivalent for  $W + X(Y + Z \cdot X')$  that omits  $Z$

**Task 29** Use Boolean algebra identities, including the Law of Absorption (15ab) to show that the following networks are equivalent.



**Task 30** This task provides some practice with networks represented by equivalent Boolean algebra expressions.

(a) Write a Boolean algebra expression for the following network:



(b) Use Boolean algebra identities to show the network given in part (a) is equivalent to the following:

$$W \cdot (XY + W') \cdot [S'YZ' + VYZ' + SWZ + XZ].$$

(c) Sketch the network for the expression  $W \cdot (XY + W') \cdot [S'YZ' + VYZ' + SWZ + XZ]$ , and comment on the relative simplicity of the two equivalent networks.

## 5 Boolean Algebras: A Modern Approach

The study of Boolean algebra continues to be an important subject, both from a theoretical point of view for mathematicians and from a practical point of view for computer scientists and engineers. For both sorts of practitioners, it is standard to start with a set of axioms that gives, in a manner of speaking, the

‘rules of the game.’ This approach is not dissimilar from the way in which the original practitioners of British Symbolic Algebra thought about their work as beginning with a set of formal laws of operation. The following modern definition of a Boolean algebra shows the standard way of doing this today.<sup>18</sup>

**Definition** A *Boolean algebra* is a six-tuple  $(S, +, \cdot, ', 0, 1)$  that satisfies the following eight axioms, where  $S$  is a set,  $+$ ,  $\cdot$  are binary operations on  $S$ ,  $'$  is a unary operation on  $S$  and  $0$ ,  $1$  are elements of  $S$ .

- |  |  |
|--|--|
| 1. Commutativity of Addition:                      | $(\forall x, y \in S)(x + y = y + x)$                                |
| 2. Commutativity of Multiplication:                | $(\forall x, y \in S)(x \cdot y = y \cdot x)$                        |
| 3. Associativity of Addition:                      | $(\forall x, y, z \in S)((x + y) + z = x + (y + z))$                 |
| 4. Associativity of Multiplication:                | $(\forall x, y, z \in S)((x \cdot y) \cdot z = x \cdot (y \cdot z))$ |
| 5. Distributivity of Multiplication over Addition: | $(\forall x, y, z \in S)(x \cdot (y + z) = x \cdot y + x \cdot z)$   |
| 6. Distributivity of Addition over Multiplication: | $(\forall x, y, z \in S)(x + y \cdot z = (x + y) \cdot (x + z))$     |
| 7. Identity Laws:                                  | $(\forall x \in S)(x + 0 = x \text{ and } x \cdot 1 = x)$            |
| 8. Complement Laws:                                | $(\forall x \in S)(x + x' = 1 \text{ and } x \cdot x' = 0)$          |

Although the Boolean algebra structure defined above is intended to be completely *abstract* in the sense that no particular meaning is ascribed to the symbols  $+$ ,  $\cdot$ , etc, there are many specific *concrete* examples of Boolean algebras of interest from both a theoretical point of view and from a practical point of view, especially in computer science and engineering. Elementary set theory — which is essentially the system that Boole himself studied — is one such concrete example. To see how this particular example satisfies the modern definition of a Boolean algebra, let  $S$  be the set of all subsets<sup>19</sup> of a given universal set  $U$ . The eight Boolean algebra axioms are then translatable into the language of set theory as follows:

- |   |   |
|---|---|
| 1. Commutativity of Union:                    | $(\forall A, B \in S)(A \cup B = B \cup A)$                               |
| 2. Commutativity of Intersection:             | $(\forall A, B \in S)(A \cap B = B \cap A)$                               |
| 3. Associativity of Union:                    | $(\forall A, B, C \in S)((A \cup B) \cup C = A \cup (B \cup C))$          |
| 4. Associativity of Intersection:             | $(\forall A, B, C \in S)((A \cap B) \cap C = A \cap (B \cap C))$          |
| 5. Distributivity of Intersection over Union: | $(\forall A, B, C \in S)(A \cap (B \cup C) = (A \cap B) \cup (A \cap C))$ |
| 6. Distributivity of Union over Intersection: | $(\forall A, B, C \in S)(A \cup (B \cap C) = (A \cup B) \cap (A \cup C))$ |
| 7. Identity Laws:                             | $(\forall A \in S)(A \cup \emptyset = A \text{ and } A \cap U = A)$       |
| 8. Complement Laws:                           | $(\forall A \in S)(A \cup A' = U \text{ and } A \cap A' = \emptyset)$     |

Because the operations of set theory do, in fact, satisfy the eight axioms that define a Boolean algebra, we can conclude that elementary set theory is a particular example of a Boolean algebra. Accordingly, it is now possible to translate any theorem from the general theory of an abstract Boolean algebra into a specific property for sets; the following table gives this translation for just a few of these theorems. (*We will consider the proofs of these theorems within an abstract Boolean algebra in Task 34.*)

<sup>18</sup>Of course, it's necessary that the given laws, or axioms, be consistent with each other. For a study of independence of the axioms for a Boolean algebra based on a primary historical source written by E. V. Huntington, see [Barnett, 2013b].

<sup>19</sup>The set  $S$  of all subsets of  $U$  is also called the ‘power set of  $U$ ’ and denoted  $\mathcal{P}(U)$ . For example, if  $U = \{a, b, c\}$ , then  $\mathcal{P}(U) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ . In general,  $|\mathcal{P}(U)| = 2^{|U|}$ , as illustrated in this example.

Theorem/Property	Boolean Algebra Version For all $x, y \in S$ :	Set Theoretic Version For all sets $A, B \subseteq U$ :
De Morgan's Laws	$(x + y)' = x'y'$ $(xy)' = x' + y'$	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
Absorption Laws	$x + xy = x$ $x(x + y) = x$	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Idempotent Laws	$x + x = x \cdot x = x$	$A \cup A = A \cap A = A$
Involution Law	$(x')' = x$	$(A')' = A$
Bound Laws	$x + 1 = 1$ $x \cdot 0 = 0$	$A \cup U = U$ $A \cap \emptyset = \emptyset$

In a similar fashion, the fundamental laws of *every* concrete example of a Boolean algebra can be obtained, a fact that explains in part why the *abstract structure* of a Boolean algebra is not only an interesting object in its own right, but also a powerful theoretical and applied tool. We conclude this project with a set of tasks in based on the above definition that further explores the modern axiomatic approach.

**Task 31** Let  $S = \{1, 2, 3, 6\}$  with the operations  $+$ ,  $\cdot$  and  $'$  defined on  $S$  as follows:

$$\text{Given } x, y \in S, \text{ let } x + y = \text{lcm}(x, y), \quad x \cdot y = \text{gcd}(x, y) \text{ and } x' = \frac{6}{x}.$$

In this task, we will give a partial proof that  $(S, +, \cdot, ', 1, 6)$  is an Boolean algebra by verifying that it satisfies certain axioms in the modern definition.

(a) Begin by completing the following tables.

$+$	1	2	3	6
1				
2				
3				
6				

$\cdot$	1	2	3	6
1				
2				
3				
6				

$x$	1	2	3	6
$x'$				

(b) The order of the elements in the six-tuple  $(S, +, \cdot, ', 1, 6)$  is important!

In particular, this order suggests that

- \* The additive identity (**0**) will be 1:  $x + 1 = x$  for all  $x$
- \* The multiplicative identity (**I**) will be 6:  $x \cdot 6 = x$  for all  $x$ .

Explain how we know that both these properties hold.

(This will give a proof of Axiom #7, the Identity Laws.)

(c) Verify that Axiom #8, the Complement Laws hold. (Remember: **0** = 1 and **I** = 6.)

**Task 32**

Before we finish the proof that the algebraic structure in the preceding task is a Boolean algebra, let's look at a similar structure that is NOT a Boolean algebra.

Let  $S = \{1, 2, 4, 8\}$  with the operations  $+$ ,  $\cdot$  and  $\prime$  defined on  $S$  as follows:

Given  $x, y \in S$ , let  $x + y = \text{lcm}(x, y)$ ,  $x \cdot y = \text{gcd}(x, y)$  and  $x' = \frac{8}{x}$ .

(a) Begin by completing the following tables.

$+$	1	2	4	8
1				
2				
4				
8				

$\cdot$	1	2	4	8
1				
2				
4				
8				

$x$	1	2	4	8
$x'$				

- (b) Show that Axiom #7, the Identity Laws, do hold for this algebra by finding the additive identity and the multiplicative identity. Explain how you know these are correct.
- (c) Show that BOTH the Complement Laws (Axiom #8) FAIL for this algebra.
- (d) The failure of the Complement Laws, which you proved in part (c) of this task, is actually all we need to conclude that this structure is NOT a Boolean algebra. But perhaps there is some other Boolean algebra property that also fails for this particular structure. Try to find one of these (either an axiom or a major property) and explain how you know that it fails, or prove that Axioms #1–6 in the definition of a Boolean algebra hold for this particular example.

**Task 33**

This task continues the proof that the algebraic structure from Task 31 is a Boolean algebra.

RECALL:  $S = \{1, 2, 3, 6\}$   $x + y = \text{lcm}(x, y)$ ,  $x \cdot y = \text{gcd}(x, y)$  and  $x' = \frac{6}{x}$ .

Also recall that Axioms #7 and #8 were proven in Task 31.

- (a) Use the definitions of multiplication and addition to explain why Axioms #1 and #2, the Commutativity Laws, are true.
- (b) Explain why Axioms #3 and #4, the Associativity Laws, are true.
- (c) Explain why Axioms #5 and #6, the Distributivity Laws, are true.

**Task 34** Let  $(S, +, \cdot, ', 0, 1)$  be a Boolean algebra.

Write a proof of each of the following properties, based on the format shown in the example below. *Be sure that each step of your proof is either an axiom or a property that has already been proven from the axioms.*

**Proof of Bound Law for Addition:** For all  $x \in S$ ,  $x + 1 = 1$ .

$x + \mathbf{I}$	$= (x + \mathbf{I}) \cdot \mathbf{I}$	by the Identity Property for Multiplication
	$= (x + 1) \cdot (x + x')$	by the Complement Property for Addition
	$= x + 1 \cdot x'$	by the Distributivity of Addition over Multiplication
	$= x + x' \cdot 1$	by Commutativity of Multiplication
	$= x + x'$	by the Identity Property for Multiplication
	$= 1$	by the Complement Property for Addition

- (a) Bound Law for Multiplication: For all  $x \in S$ ,  $x \cdot 0 = 0$
- (b) Absorption Law for Addition: For all  $x, y \in S$ ,  $x + xy = x$
- (c) Absorption Law for Multiplication: For all  $x, y \in S$ ,  $x(x + y) = x$
- (d) Idempotent Law for Multiplication: For all  $x \in S$ ,  $xx = x$
- (e) Idempotent Law for Addition: For all  $x \in S$ ,  $x + x = x$
- (f) For all  $x, y \in S$ , if  $x + y = x + z$  and  $x' + y = x' + z$ , then  $y = z$ .
- (g) For all  $x, y \in S$ , if  $x + y = x + z$  and  $xy = xz$ , then  $y = z$ .
- (h) Uniqueness of Complements: For all  $x, y \in S$ , if  $x + y = \mathbf{I}$  and  $xy = 0$ , then  $y = x'$ .

**Task 35** It turns out that it is possible to prove associativity from the other axioms. Do this! That is, show that the Associativity Axioms are actually theorems by proving them from the remaining axioms.

NOTE: This proof is not completely straightforward, which is why most textbooks simply list associativity as an axiom. Of course, including it as an axiom means we have two more conditions to verify in order to show that a given algebra is, in fact, a Boolean algebra.

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## Notes to the Instructor

### PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed for an introductory or intermediate course in discrete or finite mathematics or in an “introduction to proof” course. Its primary goal is to introduce students to a more abstract view of algebra, one focused on the structure and underlying axioms of a system, through an examination of the specific structure known today as a Boolean algebra. It does this through historical examples, including select readings from two important primary sources: George Boole’s 1847 *Laws of Thought* and Claude Shannon’s 1938 *A Symbolic Analysis of Relay and Switching Circuits*. Both sources also raise questions related to mathematical proof that are further and more explicitly addressed within the final non-historical section of the project on modern Boolean algebra.

Although Boole’s writing is essentially focused on what would now be called ‘introductory set theory,’ it also lays the groundwork for a more abstract discussion of Boolean algebra as a discrete axiomatized structure. Importantly for students making the transition to more abstract mathematical studies, it does so in a very concrete fashion, while simultaneously exposing what seem like strange algebraic rules such as idempotency. Shannon’s paper is similarly quite concrete, and has the advantage of revealing the important applied side of Boolean algebra. Accordingly, this project may also be used as a complete introduction to the study of elementary Boolean algebra in any course that considers that structure from either a mathematical or computer science perspective.

The various sections of the present PSP are largely extracted from several other PSPs (with the permission of their author), including [Barnett, 2011], [Barnett, 2013a] and [Barnett, 2013c]. See the subsection “Connections to other Primary Source Projects” of these Notes below for further information on these and other projects that could be used to address aspects of elementary set theory and/or Boolean algebra in junior-level courses aimed at either mathematics or computer science courses.

### Student Prerequisites

Beyond a certain level of mathematical maturity, commensurate with a typical Calculus I background, there are no pre-requisites for this project. Strong students at a pre-calculus level have also successfully completed portions of this project.

### PSP Design

This PSP begins with a short (un-numbered) historical introductory section that describes the mathematical context in which George Boole developed his ground-breaking approach to logic that eventually led to the structure that now bears his name. Of central importance to that context was the subject now known as British Symbolic Algebra. Sections 1 and 2 of the project examine two specific examples of symbolic algebras, quaternions and octaves, respectively, as a means to draw students into the study of algebraic laws from a more abstract perspective. (Neither of these number systems is a Boolean algebra, as each has just one operation (i.e., multiplication).) These sections also make use of the familiar tool of a multiplication table — or a Cayley table, as it has become known in abstract algebra — to gently ease students into these ideas.

Section 3 of the PSP then moves to Boole’s writings on the use of symbolic algebra to represent logical classes, aided by the operations of logical addition (i.e., set union), logical multiplication (i.e., set intersection) and logical difference (i.e., set difference). Certain restrictions placed on the use of addition and subtraction by Boole that have since been lifted are lightly touched upon in this section. The

focus, however, is on the basic algebraic laws that govern the three operations. As they were developed and justified by Boole, the justifications presented in this section rely in part on his definitions of the operations and in part on the analogy of his symbols with those of ‘standard algebra.’

Section 4 of the PSP fast forwards about one hundred years to selections from Claude Shannon’s award-winning paper *A Symbolic Analysis of Relay and Switching Circuits*. The focus of this section is to introduce and develop the use of boolean expressions to represent parallel and series circuits. Within the concrete context of the two-valued Boolean algebra associated with these circuits, the standard properties of a Boolean algebra are developed, with specific project tasks providing students with practice in using these identities to simplify and manipulate boolean expressions. The use of circuits to justify the laws that govern a Boolean algebra is also explored. Edward Huntington’s work on the axiomatization of Boolean algebras is also briefly mentioned in this section, in part to emphasize to students the relationship between abstract axiomatic structures and concrete models as examples of those structures.

The project concludes (Section 5) by summarizing how Boole’s ‘Algebra of Logic’ relates to elementary set theory as it is typically presented today, and discussing how elementary set theory (when viewed as an algebraic structure) serves as a concrete example of a Boolean algebra. Standard (undergraduate) notation and axioms for a Boolean algebra are introduced in and used throughout this section. This section also includes a set of tasks based on the modern definition of a Boolean algebra that further explores the modern axiomatic approach, including formal proofs of several theorems that hold for that structure that were justified in less formal ways in the Boole and Shannon sections of the project.

## Suggestions for Classroom Implementation

To reap the full mathematical benefits offered by this PSP, students should be required to read assigned sections in advance of any in-class discussion, or to work through reading excerpts together in small groups in class. The author’s method of ensuring that advance reading takes place is to require student completion of daily “Reading Guides” based on the assigned reading for the next class meeting. In addition to supporting students’ advance preparation efforts, these guides provide helpful feedback to the instructor about individual and whole class understanding of the material. A typical guide includes a few “Classroom Preparation” exercises (drawn from the PSP Tasks) for students to complete prior to arriving in class; they may also include “Discussion Questions” that ask students only to read a given task and jot down some notes in preparation for class discussion.<sup>20</sup> On occasion, tasks are also assigned as follow-up to a prior class discussion. Students can also be encouraged to record any questions or comments they have about the assigned reading on their guide, or explicitly prompted to write one to three questions or comments about a particular primary source excerpt; responses to the latter type of prompt can be especially useful as starting points for in-class discussions and as feedback to the instructor. In order to incorporate advance preparation into course grades, reading guides are collected each class period for instructor review and scoring prior to the next class period. The author’s students receive credit based only on completion (with no penalty for errors in solutions).

With regard to implementation, a combination of small-group work, whole-class discussions, student presentations and homework assignments drawn from the PSP tasks is recommended in order to take

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<sup>20</sup>Experience has proven the value of reproducing the full text of any assigned project task on the guide itself, with blank space deliberately left below each question for the student response. This not only makes it easier for students to record their thoughts as they read, but also makes their notes more readily available to them during in-class discussions and easier for the instructor to efficiently review for completeness or a quick assessment of students’ understanding.

advantage of the variety of questions provided in the PSP. The Sample Implementation Schedule below includes suggestions for tasks especially well-suited to these different instructional strategies. For small-group work on individual tasks, the author recommends providing students with a copy of the task (with space provided to complete each part thereof). L<sup>A</sup>T<sub>E</sub>X code of the entire PSP may be requested from the author to facilitate preparation of such ‘in-class task sheets’ and/or advance Reading Guides.

### Sample Implementation Schedule (based on a 50-minute class period)

The following six-day sample schedule assumes completion of the entire PSP in about two weeks of class time. The recommended small-group discussion/work time should naturally be supplemented with whole-class discussions as deemed appropriate by the instructor.

- **Advance Preparation Work for Day 1** (to be completed before class)  
Read the (un-numbered) introduction and the start of Section 1 (through Task 2), and complete Tasks 1–2 for class discussion.
- **Day 1 of Class Work** Whole class and/or small group discussion of the following:
  - (Optional) Historical and mathematical ideas from the introduction, if desired.
  - Mathematical concepts in Section 1, including answers to Tasks 1–2.
  - Complete Section 1 by working Tasks 3–5.
  - Complete all of Section 2, including Tasks 6–8.
- **Advance Preparation Work for Day 2** (to be completed before class)  
Read the start of Section 3 (through Task 10), and complete Tasks 9–10 for class discussion.
- **Day 2 of Class Work** Whole class and/or small group discussion of the following:
  - Mathematical concepts in assigned reading from Section 3, including answers to Tasks 9–10.
  - Continue Section 3, ending with Task 13. As needed, Task 13 could also be added to the advanced preparation work for Day 3.
- **Advance Preparation Work for Day 3** (to be completed before class)  
Continue reading of Section 3 (through Task 15), and complete Tasks 14–15 for class discussion.
- **Day 3 of Class Work** Whole class and/or small group discussion of the following:
  - Mathematical concepts in assigned reading from Section 3, including answers to Tasks 14–15.
  - Complete Section 3, ending with Task 18 and possibly assigning some tasks (e.g., Tasks 16, 17, and/or 18) in this section as homework or as advanced preparation work for Day 4.
- **Advance Preparation Work for Day 4** (to be completed before class)  
Begin reading Section 4 (through Task 19), and complete Task 19 for class discussion.
- **Day 4 of Class Work** Whole class and/or small group discussion of the following:
  - Mathematical concepts in assigned reading from Section 4, including answers to Task 19.
  - Continue Section 4, ending with Task 25. A whole-class discussion of parts (c) and (d) of Task 25 is desirable, but could be saved until Day 5. Task 26 should be saved for the final homework assignment (possibly started in class if time permits).

- **Advance Preparation Work for Day 5** (to be completed before class)  
Continue reading of Section 4 (through Task 28), and complete Task 27(a) and Task 28 for class discussion.
- **Day 5 of Class Work** Whole class and/or small group discussion of the following:
  - Mathematical concepts in assigned reading from Section 4, including answers to Task 27(a) and Task 28. Parts (c) and (d) of Task 27 can be saved for the final homework assignment (possibly started in class if time permits).
  - Continue Section 4, ending with Task 30. Task 27 should be saved for the final homework assignment (possibly started in class if time permits).
- **Advance Preparation Work for Day 6** (to be completed before class)  
Read the start of Section 5 (through Task 31), and complete Task 31 for class discussion.
- **Day 6 of Class Work** Whole class and/or small group discussion of the following:
  - Mathematical concepts in assigned reading from Section 5, including answers to Task 31 as needed.
  - Continue Section 5, ending with Task 34(b) or Task 34(c). The remainder of Task 34 should be saved for the final homework assignment.
- **Final Homework Assignment** – following completion of all in-class work, a complete formal write-up of the following should be assigned: Tasks 5, 8, 11, 14, 16, 17, 18, 26, 27(b,c), 33, 34 (some or all parts). Task 35 can also be assigned either as required homework or for extra credit only.

### Connections to other Primary Source Projects (PSPs)

Instructors who wish to offer a more intensive primary source based exploration of topics related to Boolean algebra will want to consider the following series of three independent PSPs, each designed for use in undergraduate courses in discrete mathematics or those intended as an introduction to proof.

- “Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce,” [Barnett, 2013b], develops a modern understanding of these operations and their basic properties within the context of early efforts to develop a symbolic algebra for logic, but without explicitly introducing modern notation for operations on sets (until the concluding section). It begins with a more extended discussion of Boole’s *Laws of Thought* than appears in the present PSP. By then following the refinements made to Boole’s symbolic algebra in the hands of Venn and Peirce, this project provides students with an opportunity to witness first-hand *how* the process of developing and refining a mathematical system plays out, the ways in which mathematicians make and explain their choices along the way, and how standards of rigor in these regards have changed over time. Thus, in addition to developing the properties of set theory as a specific concrete example of a Boolean algebra, the project is able to explore a variety of mathematical themes, including the notion of an inverse operation, the concept of duality, issues related to mathematical notation, and standards of rigor and proof. By following one or more of these themes through the project, instructors have considerable leeway in tailoring the project to their goals for a particular group of students. By steadily increasing the level of abstraction, that project also lays the ground work for a more abstract discussion of a Boolean algebra as a discrete structure.

- “Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization,” [Barnett, 2013b], explores the early axiomatization of Boolean algebra as an abstract structure through readings from Huntington’s 1904 paper *Sets of Independent Postulates for the Algebra of Logic*. In addition to introducing the now standard axioms for the Boolean algebra structure, the project illustrates how to use these postulates to prove some basic properties of Boolean algebras. Specific project tasks also provide students with practice in using symbolic notation, and encourage them to analyze the logical structure of quantified statements. The project also examines Huntington’s use of the two-valued Boolean algebra on  $K = \{0, 1\}$  — first studied by George Boole in his work on the logic of classes — as a model to establish the *independence* and *consistency* of one of his postulate sets. The final section of the project discusses modern (undergraduate) notation and axioms for Boolean algebras, and provides several practice exercises to reinforce the ideas developed in the earlier sections.
- “Applying Boolean Algebra to Circuit Design: Claude Shannon,” [Barnett, 2013a], based on Shannon’s ground-breaking paper *A Symbolic Analysis of Relay and Switching Circuits*, begins with a concise overview of the two major historical antecedents to Shannon’s work: Boole’s original work in logic and Huntington’s work on axiomatization. The project then develops standard properties of a Boolean algebra within the concrete context of circuits, and provides students with practice in using these properties to simplify boolean expressions. The two-valued Boolean algebra on  $K = \{0, 1\}$  again plays a central role in this work. The project closes with an exploration of the concept of a ‘disjunctive normal form’ for boolean expressions, again within the context of circuits.

The above three projects are available at [www.cs.nmsu.edu/historical-projects/projects.php](http://www.cs.nmsu.edu/historical-projects/projects.php) or upon request to the author. They may be used either alone or in any combination with each other depending on an instructor’s goal. **The second of these, “Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization” [Barnett, 2013b], is recommended as well as a follow-up to the present PSP for use in courses that aim to extend students’ understanding of formal axiomatization and proof.**

The following additional PSPs, while of a more advanced nature than those listed above, address core topics from the standard curriculum of a junior-level abstract algebra course. Each has been successfully site tested at several institutions as a replacement for a textbook, either for a portion of the course, or for the course in its entirety. Further information about structuring an entire Abstract Algebra course around PSPs in this collection is available from the author of these projects.

- “Abstract Awakenings in Group Theory: Early group theory in the works of Lagrange, Cauchy, and Cayley”<sup>21</sup>

The centerpiece of this extended PSP is the 1854 inaugural paper on abstract group, Arthur Cayley’s “On the theory of groups, as depending on the symbolic equation  $\theta^n = 1$ ” [Cayley, 1854]. In keeping with the historical record, and to provide concrete examples on which to base their abstraction of the group concept, Section 1 of the project begins with the material from Lagrange in the PSP “The Roots of Early Group Theory in the Works of Lagrange” (described below). Section 2 then employs selections from writings by Cauchy in the PSP “An Independent Theory of Permutations: Early Group Theory in the Work of A.-L. Cauchy” (also described below). The Abstract “Awakenings” project then turns to a detailed reading of Cayley’s complete paper in Sections 3 and 4, paying careful attention to the similarities between the theory of permutation groups as it was developed by Cauchy and the modern notion of an abstract group as it was unveiled by Cayley.

Absolutely no familiarity with group theory is assumed in this PSP! Instead, it was explicitly designed to serve as students’ very first encounter with group-related ideas. Completion of the entire project takes approximately 10 weeks, but (un)covers the vast majority of the elementary group theory typically studied in a junior level abstract algebra course, including: roots of unity, permutations, definition and elementary properties of group (including results related to the order of group elements), abelian groups, cyclic groups, symmetric groups, alternating groups, Cayley tables, Lagrange’s Theorem, group isomorphisms, classification of groups of small order, and direct products. The concept of cosets are also introduced in the main body of the project, and further developed in an appendix that also states the definitions of normal subgroup and factor group; this material is, however, more fully and effectively developed in the Höder Quotient Group PSP described below.

- “The Roots of Early Group Theory in the Works of Lagrange”<sup>22</sup>

This PSP draws on works by one of the early precursors of abstract group, French mathematician J. L. Lagrange. An important figure in the development of group theory, Lagrange made the first real advance in the problem of solving polynomial equations by radicals since the work of Cardano and his sixteenth-century contemporaries. In particular, Lagrange was the first to suggest the existence of a relation between permutations and the solution of equations by radicals, a suggestion later exploited by Abel and Galois. In addition to the important group-theoretic concept of a permutation, the project employs excerpts from Lagrange’s study of roots of unity to develop the concept of a finite cyclic group. Lagrange’s description of his quest for a general method of algebraically solving all polynomial equations is also a model of mathematical research that make him a master well worth reading by today’s students of mathematics.

The design of this project is based on the first section of the extended PSP “Abstract Awakenings in Group Theory” described above. Instructors who begin their study of group theory with the PSP

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<sup>21</sup>To obtain the most recent version of “Abstract Awakenings in Group Theory”, contact its author at [janet.barnett@csupueblo.edu](mailto:janet.barnett@csupueblo.edu), or visit [www.cs.nmsu.edu/historical-projects/projects.php](http://www.cs.nmsu.edu/historical-projects/projects.php) for an earlier version.

<sup>22</sup>To obtain the most recent version of “The Roots of Early Group Theory in the Works of Lagrange”, visit <https://blogs.ursinus.edu/triumphs/>. An alternative version of this PSP that adopts a more open-ended/inquiry-based approach in which all resolvent equation examples are presented as tasks for students to complete themselves is also available upon request from its author at [janet.barnett@csupueblo.edu](mailto:janet.barnett@csupueblo.edu).

“The Roots of Early Group Theory in the Works of Lagrange” and then wish to continue with the pedagogy of primary source projects throughout their students’ study of group theory could easily shift over to the PSP “Abstract Awakenings of Algebra”. For those who prefer a less extended use of this instructional practice, the PSP “The Roots of Early Group Theory in the Works of Lagrange” could also be used in conjunction with a more traditional textbook. In either case, this PSP will be more effective as an exploratory introduction to the group concept if it is used *before* students have studied the concepts of cyclic groups in much, if any, detail.

- “An Independent Theory of Permutations: Early Group Theory in the Work of A.-L. Cauchy”<sup>23</sup>

This PSP draws on works by Augustin-Louis Cauchy to introduce aspects of the theory of finite groups within the concrete context of permutation groups. It consists of two main sections, the first of which focuses on the multiplication of permutations and its properties, including decomposition into disjoint cycles and even/odd permutations. The second section of the project brings in the definition and elementary theory of permutation groups (‘system of conjugate permutations’ in Cauchy’s terminology; its main focus is a detailed study of Cauchy’s proof of Lagrange’s Theorem for groups of permutations. The concrete language of that proof allows students to develop an understanding of its meaning without becoming lost in the abstraction of cosets, partitions and equivalence relations, while its complete generalizability of the proof to any finite group prepares them to make the transition to that level of abstraction later in the project/course. Modern notation for  $S_n$  and  $A_n$  are introduced.

The design of this project is based on the second section of the extended PSP “Abstract Awakenings in Group Theory” described above. Instructors who begin their study of group theory with the PSP “An Independent Theory of Permutations: Early Group Theory in the Work of A.-L. Cauchy” and then wish to continue with the pedagogy of primary source projects throughout their students’ study of the abstract group concept could shift relatively easily over to the PSP “Abstract Awakenings of Algebra”, perhaps beginning that project with its third section (on Cayley). For those who prefer a less extended use of this instructional practice, the PSP “An Independent Theory of Permutations: Early Group Theory in the Work of A.-L. Cauchy” could also be used in conjunction with a more traditional textbook. In either case, this PSP will be more effective as an exploratory introduction to the group concept if it is used *before* students have studied the concept of permutations and the symmetric group in much, if any, detail.

- “Otto Hölder’s Formal Christening of the Quotient Group Concept”<sup>24</sup>

This PSP draws on excerpts from the paper by Otto Hölder in which he gave what is now considered to be the first “modern” definition of quotient groups. Although quotient groups implicitly appeared in Galois’ work on algebraic solvability in the 1830s, that work itself pre-dated the development of an abstract group concept. Even Cayley’s 1854 paper in which a definition of an abstract group first appeared was premature, and went essentially ignored by mathematicians for decades. Permutation

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<sup>23</sup>For the most recent version of the PSP “An Independent Theory of Permutations: Early Group Theory in the Work of A.-L. Cauchy,” see *The Annals of the TRIUMPHS Society* (<https://triumphssociety.org/annals>), Volume 1, Issue 1.

<sup>24</sup>To obtain the most recent version of ‘Otto Hölder’s Formal Christening of the Quotient Group Concept’, visit <https://blogs.ursinus.edu/triumphs/>

groups were extensively studied during that time, however, with implicit uses of quotient groups naturally arising within it. Jordan, for example, used the idea of congruence of group elements modulo a subgroup to produce a quotient group structure. Thus, when Hölder wrote his paper in 1889, he was able to treat the concept as neither new nor difficult. This PSP, designed for a first course in abstract algebra, draws on excerpts from that paper as a means to introduce students to the concepts of a normal subgroup, a quotient group, the Fundamental Homomorphism Theorem and related elementary results. Excerpts from earlier works by Cauchy, Cayley and Jordan in which precursors of these ideas appeared are also treated in three optional and independent appendices.

No prior familiarity with normal subgroups, quotient groups, or group homomorphisms is assumed in this project. To the contrary, the project is designed to serve as students' first introduction to these three concepts and their related theory, following their study of more elementary group theory. It is assumed that students are comfortable with the definitions and examples of groups and subgroups, along with related proof techniques (e.g., for establishing closure under products) and basic results (e.g., Lagrange's Theorem for finite groups). Although the concept of a coset also naturally makes an appearance in this project, the definition given in the project could serve as students' first introduction to this concept. In particular, it is not necessary for students to have seen a proof of Lagrange's Theorem via cosets and equivalence classes; an alternate proof of this theorem that uses neither of these notions is included in Appendix I of the project. In addition to being fully self-contained with respect to the study of group homomorphisms, the project's treatment of the Fundamental Homomorphism Theorem also requires no prior study of group isomorphisms. It is, however, standard (and helpful!) for students enrolled in an abstract algebra course to have previously met the idea of an isomorphism in a linear algebra course.

- “Richard Dedekind and the Creation of an Ideal: Early Developments in Ring Theory”<sup>25</sup>

This PSP draws on the 1877 version of Dedekind's theory of ideals as a means to introduce students to the elementary theory of rings and ideals. Characteristics of Dedekind's work that make it an excellent vehicle for students in a first course on abstract algebra include his emphasis on abstraction, his continual quest for generality and his careful methodology. The 1877 version of his ideal theory (the third of four versions he developed in all) is an especially good choice for students to read, due to the care Dedekind devoted therein to motivating why ideals are of interest to mathematicians by way of examples from number theory that are readily accessible to students at this level. In this regard, unique prime factorization (and the failure thereof in certain integral domains) plays a central role. Other specific topics developed in the PSP include the following: rings, integral domains, fields, zero divisors, ideals, principal ideals, prime ideals, maximal ideals.

No prior familiarity with ring theory is assumed. The project has also been successfully used with students who had not yet studied group theory. For those who have not yet studied group theory (or those who have forgotten it!), basic definitions and results about identities, inverses and subgroups are fully stated when they are first used within the PSP (with the minor exception of Lagrange's Theorem for Finite Groups that is needed for one part of one task).

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<sup>25</sup>To obtain the most recent version of “Richard Dedekind and the Creation of an Ideal: Early Developments in Ring Theory”, visit <https://blogs.ursinus.edu/triumphs/>.

Additionally, the PSP “Early Writings on Graph Theory: Hamiltonian Circuits and The Icosian Game” contains two sub-sections, entitled “The Icosian Game and Hamiltonian Circuits” and “The Icosian Game and Non-Commutative Algebra,” respectively, both of which were developed specifically for use in an introductory undergraduate course in discrete mathematics. However, because no prior background in graph theory is assumed, the connection to symbolic algebra makes the project suitable for use in either a junior-level abstract algebra course, or in an introduction to proof course that emphasizes algebraic concepts. As mentioned in the present PSP, several new number systems were discovered by various British algebraists following Hamilton’s 1843 announcement of his discovery of quaternions. In his ‘Icosian Game’, Hamilton himself developed yet another example of a non-commutative algebraic system. The PSP “Hamiltonian Circuits and The Icosian Game” explores both the algebra of that system and the graph theoretical notion of ‘Hamiltonian circuit’ on which Hamilton’s interpretation of this algebra is based. The idea for the game was first exhibited by Hamilton at an 1857 meeting of the British Association in Dublin, and later sold for 25 pounds to ‘John Jacques and Son,’ a wholesale dealer in games. Excerpts from the instruction pamphlet that accompanied the game and from the British Association meeting report provide the primary sources used in this PSP. A published version of the project appears in *Resources for Teaching Discrete Mathematics*, editor B. Hopkins, Washington, DC: Mathematical Association of America, pp. 217–223. To obtain a more recent version, contact its author at [janet.barnett@csupueblo.edu](mailto:janet.barnett@csupueblo.edu).

## Recommendations for Further Reading

Instructors who wish to take a deeper dive into the project’s context or content during its classroom implementation will find the following to be of interest:

- Bocheński, I. M. (1961). *A History of Formal Logic*. University of Notre Dame Press, Notre Dame. Translated and edited by I. Thomas.
- Richards, J. L. (1980). The Art and the Science of British Algebra: A Study in the Perception of Mathematical Truth. *Historia Mathematica*, 7:353–365.
- Richards, J. L. (1992). God, Truth, and Mathematics in Nineteenth Century England. In Nye, M. J., Richards, J. L., and Stuewer, R., editors, *The Invention of Physcial Science: Intersections of Mathematics, Theology and Natural Philosophy Since the Seventeenth Century Essays in Honor of Erwin N. Hiebert*, pages 51–78. Springer.

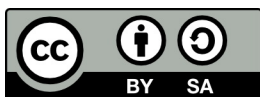
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For more information about Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit <https://blogs.ursinus.edu/triumphs/>.



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