

Archimedes' *Measurement of a Circle*

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December 5, 2025

If one were to form a “dream team” of mathematicians throughout history, it would be difficult not to strongly consider Archimedes of Syracuse (c. 287 BCE–c. 212 BCE) for one of the starting positions. Archimedes’ contributions to mathematics, physics, engineering, and astronomy have profoundly influenced mathematical and scientific thought for more than two thousand years. A partial list of his original works in Euclidean geometry includes an approximation to π ; the area of a circle; the surface area and volume of a sphere; the area of an ellipse; and the area bounded by a parabola and a secant line. Indeed, Archimedes devised techniques that anticipated the principles of calculus and analysis, using the method of exhaustion: repeatedly approximating a region by simpler ones and reducing the remaining error step by step. The legacy associated with the life (and dramatic death) of Archimedes has been the subject of writers and artists alike. For information on Archimedes’ life and writings, consult the MacTutor History of Mathematics archive, which offers an extensive collection of mathematician biographies (Robertson and O’Connor, 2003). For a more detailed look, a great place to start is William Dunham’s excellent book, *Journey Through Genius: The Great Theorems of Mathematics* (Dunham, 1991).

Archimedes’ *Kýklou métrēsis* (*Measurement of a Circle*) gave one of the earliest methods of approximating the value of π with great accuracy by describing an iterative process using inscribed and circumscribed polygons within and around the circle based on the angle bisector theorem. Beginning with two regular hexagons—one inscribed and one circumscribed—he performed four iterations of angle bisections to generate inscribed and circumscribed 96-gons. Based on these, he concluded: “Thus, the ratio of the circumference to the diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$ ” (Heath, 2002, p.98). In doing so, he also presented a method that could be continued to determine even more precise approximations to π . We will utilize Archimedes’ approach for generating the polygons and the ratios between their perimeters and the diameter of the circle to arrive at his estimates given above (Bonsangue, 2016). In doing so, we will try to identify steps that have never been—and may never be—fully explained, as well as acknowledge the fearlessness of Archimedes’ approach.

The primary source excerpts for this project are taken from *The Works of Archimedes*, published in 2002 by Dover Books (Heath, 2002). It is an unabridged replication of the edition published by Cambridge University Press in 1897, edited by Sir Thomas Heath. Heath (1861–1940) was a British mathematician, classical scholar, and historian of ancient Greek mathematics best known for his authoritative English translations and commentaries on Euclid, Archimedes, and Apollonius (note that the theorem of Euclid in the following section is also taken from a Heath translation). The source contains both Heath’s translation of Archimedes’ work, as well as his own commentary. Heath himself was working from a variety of sources and translations, as he described in the introduction.

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There is an important notational convention of which the reader should be aware: the passages in the excerpts enclosed by square brackets are Heath’s added commentary. Heath inserted these expressions to show intermediate steps so that the reader can better follow Archimedes’ reasoning. Although modern mathematicians are used to seeing parentheses, brackets, and braces to indicate the order of operations, this is merely a convention. In this text, Heath used square brackets to denote an expression he has added. He also occasionally used parenthesis inside of these additions to indicate order of operations. Otherwise, this text leaves the order of operations to the reader to infer from the context. As you read the following primary source excerpts, notice when square brackets are used and consider why Heath added these expressions where he did.

1 Overview of Archimedes’ Approach

The ratio of the lengths of a circle’s circumference to its diameter is the definition of π . One way to visualize this relationship is to think of wrapping a “belt” around a circle of unit diameter, as shown in **Figure 1** by using a web applet (Wade, 2025).

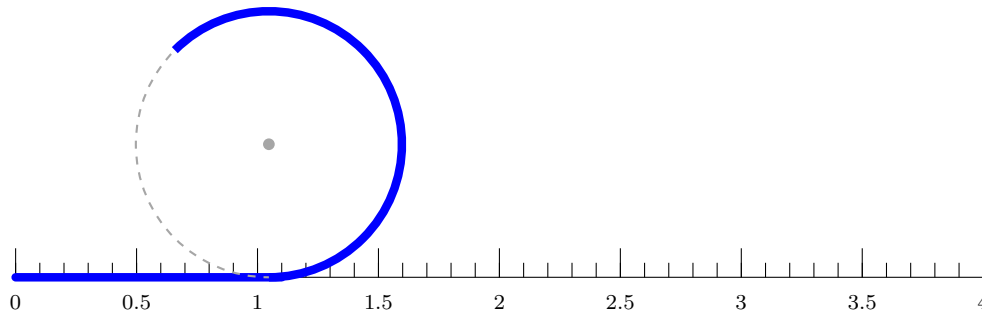


Figure 1: Unwrapping A Circle

Task 1

Unwrapping a Circle. This task allows us to gather some basic geometric intuition about the number (or ratio) π as a warm-up for our pending Archimedean journey!

- Follow the hyperlink for the web applet “**Pi Demo.**” Set the diameter of the circle to be 1. Re-create **Figure 1** and practice “wrapping” and “unwrapping” the circle. What do you notice and wonder?
- Argue that the value of π must be greater than 2 but less than 4.
- In **Figure 1**, label the center of the circle point O . Draw the radius from O to the upper-left endpoint of the unwrapped part of the circle and call it point A . Now draw the radius from O to the tangent (number) line and call it point B . Suppose the approximate number line decimal value of point B is 1.047. What is the approximate measure of $\angle AOB$? If $m\angle AOB = 120^\circ$, what would be the exact number line value for point B ?

Archimedes’ approach was to utilize circumscribed and inscribed regular polygons to create upper and lower bounds for π . He did this by comparing the perimeter of each polygon to the diameter of the circle.

Figure 2 shows Archimedes’ first step using a regular hexagon. The circles are the same size. Using the circumscribed hexagon on the left, a (rough) upper bound for π can found by taking

$6 \times JK : 2 \times GL$, a *ratio* of the perimeter of the hexagon to the diameter of the circle. Similarly, in the inscribed hexagon on the right, a lower bound can be found by taking $6 \times UT : 2 \times HU$. In other words, imagine wrapping a “belt” around each *hexagon* and comparing it as a ratio to the diameter of the circle. Archimedes continued this process for the regular 12-gon, 24-gon, 48-gon, and 96-gon to arrive at his upper and lower bounds for the value of π .

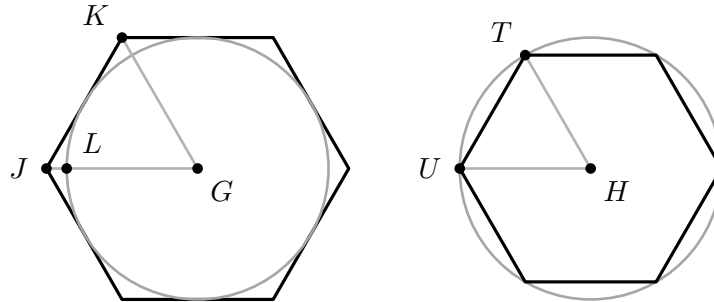


Figure 2: Circumscribed and Inscribed Hexagons

We add two notes here that may be helpful moving forward. First, Archimedes clearly assumed that the perimeter of the circumscribed polygon is greater than the circumference of the circle (and that the perimeter of the inscribed polygon is less than the circumference of the circle). Indeed, he formally stated this as Assumption 2 in his work, *Perì sphaíras kai kylíndrou* (*On the Sphere and the Cylinder*) (Heath, 2002). And second, none of the lengths used by Archimedes was absolute; rather, his lengths were always (though sometimes implicitly) expressed as the ratio of two (or more) quantities.

It is also important to understand that at the time of Archimedes, the existence of numbers which were not commensurate with natural numbers (which we now refer to as “irrational”) was a nascent concept. The notion that some quantities could not be expressed as the ratio of two whole numbers was generally held in suspicion. Numbers such as $\sqrt{3}$, which we now know cannot be expressed as the ratio of two integers, were approximated with rational numbers. Consider, for example, **Figure 3**, which shows a $30^\circ - 60^\circ - 90^\circ$ triangle ABC with squares constructed on each side. The areas of the squares on sides CB , AB , and AC are in the ratio of $4 : 3 : 1$. Thus, we would today say that the ratio of the lengths of sides BC , AB , and AC of triangle ABC is $2 : \sqrt{3} : 1$. However, using rational numbers, how would we quantify the ratio $AB : AC$? This was the starting point for Archimedes.

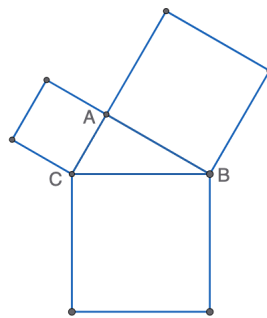
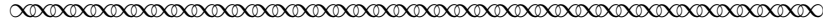
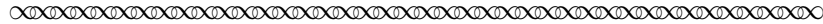


Figure 3: The $30^\circ - 60^\circ - 90^\circ$ Right Triangle

We begin by proving the angle bisector theorem (Euclid VI.3), since it formed the basis for Archimedes' approach.¹



If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle



— *Euclid, Elements VI.3* (Heath, 1956)

Task 2

Prove the Angle Bisector Theorem, Euclid VI.3. Supply the reason for each statement in the proof below. Reference **Figure 4**.

Given: Let triangle ABC be given with AD the bisector of angle BAC .

Prove: $BA : AC = BD : DC$

| Statement | Reason |
|---|---------------------------------------|
| 1. Angle $i =$ angle ii | 1. |
| 2. Let EC be parallel to AD meeting ray BA at E . | 2. |
| 3. $BA : AE = BD : DC$ | 3. <i>Hint: Reference Euclid VI.2</i> |
| 4. Angle $i =$ angle iii | 4. |
| 5. Angle $ii =$ angle iv | 5. |
| 6. Angle $iii =$ angle iv | 6. |
| 7. $AE = AC$ | 7. |
| 8. $BA : AC = BD : DC$ | 8. |

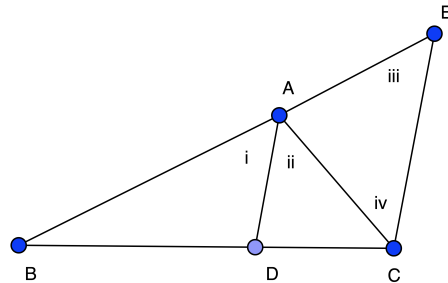
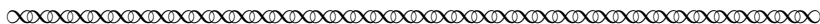


Figure 4: The Angle Bisector Theorem

¹Euclid of Alexandria (c. 300 BCE) wrote *The Elements*, a collection of thirteen books on plane and solid geometry that is widely regarded as one of the most influential works in mathematics and science ever written (Heath, 1956). Euclid VI.3 refers to *The Elements*, Book 6, Proposition 3. As was noted about Archimedes, a quick MacTutor search will turn up much information on Euclid, and, as before, Dunham (Dunham, 1991) will provide even further context and content.

2 The Circumscribed Polygons

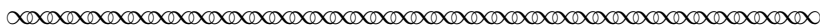
Archimedes began by circumscribing a regular hexagon around a circle, as shown in **Figure 5**.



- I. Let AB be the diameter of any circle, O its centre, AC the tangent at A ; and let the angle AOC be one-third of a right angle. Then

$$OA : AC [= \sqrt{3} : 1] > 265 : 153, \quad (1)$$

$$\text{and } OC : CA [= 2 : 1] = 306 : 153. \quad (2)$$



Task 3

The Circumscribed Hexagon. In this task, we see just how much information about π can be deduced solely from this simple construction.

- Consider a regular hexagon circumscribed about a circle (**Figure 5**). What are the measures of the three angles of triangle AOC ? How do you know?
- Using modern square root notation, what is the exact ratio of the longer leg AO to the shorter leg AC ? Hypotenuse OC to the shorter leg CA ? (Note that the modern square root notation in the excerpt is in square brackets, which means that it was added by T.L. Heath).
- Archimedes approximated the ratios of $AO : AC$ and $OC : AC$ as $265 : 153$ and $306 : 153$, respectively. Use a calculator to determine how close these ratios are to the exact ratios. Why would Archimedes choose to express $OC : AC$ as $306 : 153$ instead of $2 : 1$?
- Please note the direction of the inequality in (1). Why do you think Archimedes started with an estimate for $OA : AC$ that was *less than* its true value? ²
- Use what we know so far to obtain a fractional upper bound for π from the circumscribed hexagon ³. Comment on how good the upper bound is at approximating π .

Before proceeding to the first bisection in Archimedes' iterative process, note the following to help. The hexagon is built from a triangle AOC , and the two key numerical approximations given in (1) and (2). These can be carried forward to the first bisection as a lower bound for the ratio of the long leg to the short leg ($AO : AC$), and a lower bound (in this instance equal to) for the ratio of the hypotenuse to the short leg ($OC : CA$).

²One wonders how Archimedes obtained and chose $265 : 153$ as his lower bound for $AO : AC$. This has been part of the efforts of historians to understand classical Greek methods of approximation, e.g., (Hutchins and Adler, 1952).

³Hint: You may want to turn the ratio in (1) upside down. For example, compare the size of $5/6$ to $3/4$. Then compare the size of $6/5$ to $4/3$. Which way does the inequality point after "flipping" the fractions?

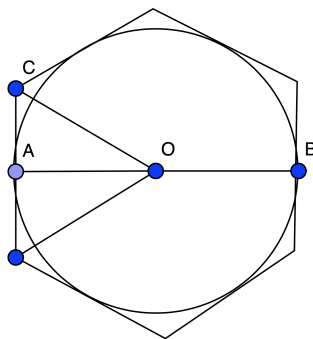
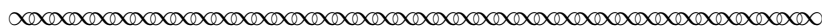


Figure 5: The Circumscribed Hexagon

Archimedes then bisected the central angle at AOC .



First, draw OD bisecting the angle AOC and meeting AC in D .

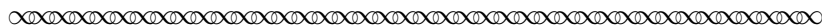
$$\text{Now } CO : OA = CD : DA, \quad (\text{Euclid VI.3})$$

so that $[CO + OA : OA = CA : DA, \text{ or}]$

$$CO + OA : CA = OA : AD.$$

Therefore [by (1) and (2)]

$$OA : AD > 571 : 153. \quad (3)$$



Task 4

The First Bisection. By bisecting this angle, Archimedes went from a circumscribed hexagon to a circumscribed 12-gon, which increased the accuracy of the ratio for approximating π . We work through the details of this approximation in this task.

- (a) Reference the triangle in **Figure 6**. Explain how Euclid VI.3 is used to obtain the first ratio $CO : OA = CD : DA$.
- (b) Add 1 to each side of the first equality of ratios above to show that $(CO + OA) : OA = CA : DA$. Rewrite this equality to get $(CO + OA) : CA = OA : DA$. Why is this true? Note that, as mentioned earlier, in Archimedes' original work parenthesis were *not* placed around the sum $CO + OA$.
- (c) Archimedes replaced the ratio of the larger leg to the smaller leg of the new triangle with the ratio of the sum of the two largest sides to the smallest side of the old triangle. Use the values in (1) and (2) to obtain the final inequality (3).
- (d) Reflect point D on ray DA about A to create point D' . Side $D'D$ represents the side of a regular polygon of how many sides circumscribed about the circle? Draw a picture.

- (e) Use (3) above to approximate π in the same way you carried out for the hexagon earlier. Notice that (3) bounds the ratio of a long leg to a short leg as discussed earlier for the hexagon. How does this approximation for π compare with that obtained using the hexagon?
- (f) (Optional Extra Credit). Calculate an *exact* formula for the perimeter of the polygon above using square roots. Then use this result to obtain a decimal approximation to π . Compare this result with that obtained in (e).

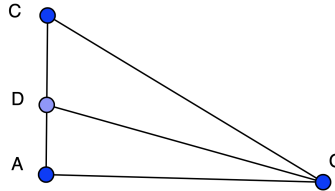
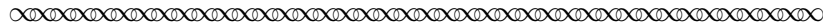


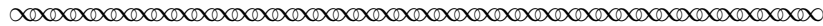
Figure 6: The First Bisection (Circumscribed Polygons)

For the next bisection, Archimedes again wanted close bounds on the ratios of the long leg to the short leg and of the hypotenuse to the short leg, this time for triangle OAD . In (3), he had the first of these ratios. We now see how he went after the second.



$$\begin{aligned} \text{Hence } OD^2 : AD^2 & [= OA^2 + AD^2 : AD^2] \\ & > (571^2 + 153^2) : 153^2 \\ & > 349450 : 23409, \end{aligned}$$

$$\text{so that } OD : DA > 591\frac{1}{8} : 153. \tag{4}$$



Task 5 **Applying the Pythagorean Theorem.** In the passage above, Archimedes used the Pythagorean Theorem to approximate the ratio of the hypotenuse to the shorter leg of triangle DAO with a closer lower bound. Again, we will follow his reasoning by working through the text.

- (a) Work through the first three lines above to arrive at $OD^2 : AD^2 > 349450 : 23409$. Again, note that the expressions enclosed by square brackets have been added by Heath. What does the ratio $349450 : 23409$ represent in terms of the areas of the geometric squares built on the sides OD and AD ?
- (b) Without explanation, Archimedes states inequality (4). Using your calculator, compare the square root of $349450/23409$ to the decimal value for $591\frac{1}{8}/153$. Why do you think Archimedes did not work with decimals?

- (a) Archimedes made choices all the way along for certain inequalities. For instance, he started with a choice of lower bound for $OA : AC$. Explain how you know this actually is a lower bound, and speculate about how he found it and how why he chose it. Archimedes also made a choice every time he calculated a ratio of hypotenuse to the shorter side. Again, speculate about how he found it and why he chose it.
- (b) Notice that the denominator 153 propagated unscathed all the way through. Explain why this happened.
- (c) **Table 1** summarizes the results of all four bisections. Each row gives a picture of the triangle, the “new” triangle after bisection, the “old” triangle before bisection, the replacement equation, the new ratio of sides, and the new ratio of hypotenuse to side. The first two rows are already filled in, as is the last column regarding the ratio of the hypotenuse to side. Please complete **Table 1**.

Important note: In **Table 1**, the labeling of the side “lengths” of the triangles in the first column needs proper interpretation. As we have observed, and as we have done throughout this lesson, Archimedes referred to side length ratios rather than absolute lengths. So, in the first triangle picture, for example, rather than say “the length of AO is 265 and the length of AC is 153,” we should say “the ratio of AO to AC is 265 to 153.” Moreover, a triangle with side length ratios $153 : 251 : 306$ is not a 30° - 60° - 90° triangle (the true ratios are, of course, $1 : \sqrt{3} : 2$). With these two observations in mind, we have found that the pictures may be very useful in visualizing the bisection process for each iteration, as well as linking the ratios of the sides based on the “old” triangle with the ratio of sides based on the “new” triangle.

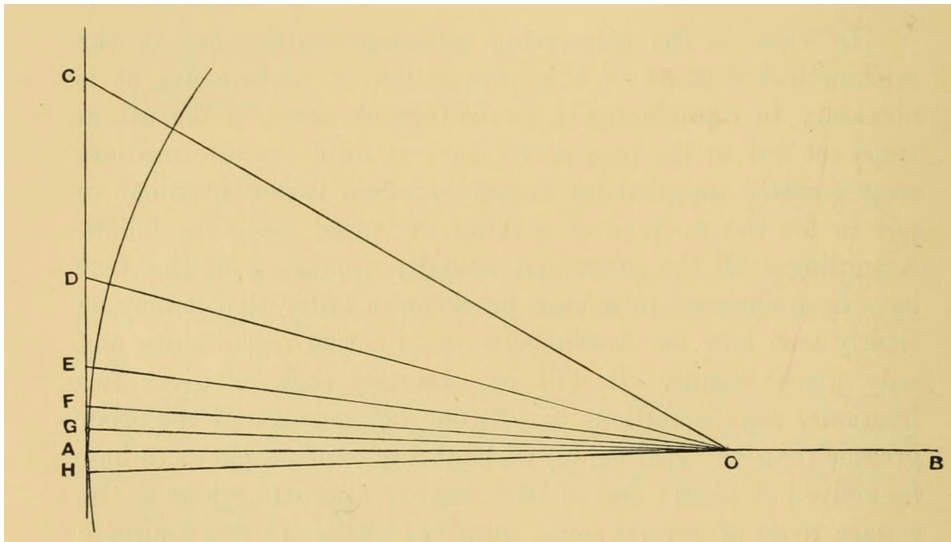


Figure 7: All Circumscribed Angle Bisections (Reprinted from (Heath, 1897))

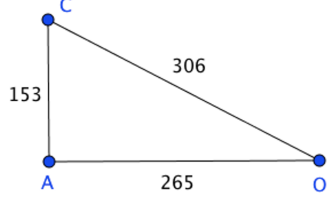
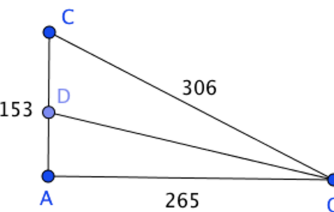
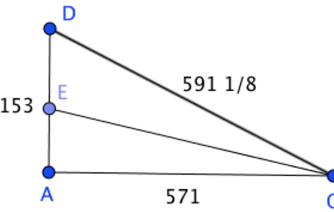
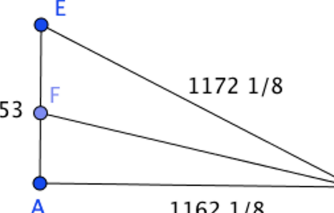
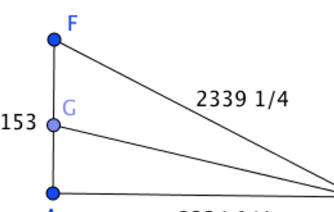
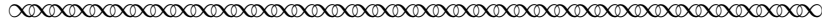
| Ratios of the Sides of the Triangle as Part of the Circumscribed Polygon of n sides | New Tri. | Old Tri. | Replacement Equation | New Ratio of Sides | New Ratio of Hyp. to Side |
|---|-----------------|-----------------|------------------------------------|--|-----------------------------------|
|  <p>Starting Triangle ($n = 6$)</p> | $\triangle CAO$ | None | None | $\frac{OA}{AC} > \frac{265}{153}$ | $CO : CA = 306 : 153 = 2 : 1$ |
|  <p>First Bisection ($n = 12$)</p> | $\triangle DAO$ | $\triangle CAO$ | $\frac{OA}{AD} = \frac{CO+OA}{AC}$ | $\frac{OA}{AD} > \frac{306 + 265}{153}$ $\frac{OA}{AD} > \frac{571}{153}$ | $DO : DA > 591\frac{1}{8} : 153$ |
|  <p>Second Bisection ($n = 24$)</p> | | | | $\frac{OA}{AE} > \frac{591\frac{1}{8}+571}{153}$ $\frac{OA}{AE} >$ | $EO : EA > 1172\frac{1}{8} : 153$ |
|  <p>Third Bisection ($n = 48$)</p> | | | | $\frac{OA}{AF} > \frac{1172\frac{1}{8}+1162\frac{1}{8}}{153}$ $\frac{OA}{AF} >$ | $FO : FA > 2339\frac{1}{4} : 153$ |
|  <p>Fourth Bisection ($n = 96$)</p> | | | | $\frac{OA}{AG} > \frac{2339\frac{1}{4}+2334\frac{1}{4}}{153}$ $\frac{OA}{AG} >$ | None |

Table 1: All Angle Bisections for the Circumscribed n -gons

Having iterated his trick, Archimedes paused the bisections in order to extract more bounds from the geometric construction.



Now the angle AOC , which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \frac{1}{48}(\text{a right angle}).$$

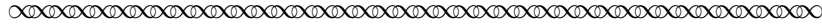
Make angle AOH on the other side of OA equal to the angle AOG , and let GA be produced meet OH in H .

Then $\angle GOH = \frac{1}{24}$ (a right angle).

Thus GH is one side of regular polygon of 96 sides circumscribed to the given circle.

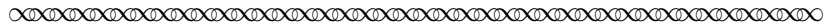
And, since $OA : AG > 4673\frac{1}{2} : 153$,
 while $AB = 2OA, GH = 2AG$,
 it follows that

$$\begin{aligned} AB : (\text{perimeter of polygon of 96 sides}) &> 4673\frac{1}{2} : 153 \times 96 \\ &> 4673\frac{1}{2} : 14688. \end{aligned}$$



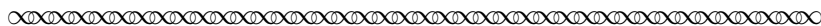
Task 8 **Creating an Upper Bound for π .** Although this process could be continued indefinitely, Archimedes used the 96-gon to establish an upper bound for π .

- (a) Per Archimedes' argument above, explain why $\angle AOG$ is $\frac{1}{48}$ of a right angle.
- (b) Explain why GH is one side of a circumscribed 96-gon.
- (c) Find a lower bound for the ratio of the diameter AB to the perimeter of the 96-gon.
- (d) Use this to find an upper bound for the ratio of the perimeter of the 96-gon to the diameter of AB . What happens to the inequality? Why? Afterwards, read Archimedes' next step, shown below, and continue with the parts of this task.



$$\begin{aligned} \text{But } \frac{14688}{4673\frac{1}{2}} &= 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} \\ &\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} \right] \\ &< 3\frac{1}{7}. \end{aligned}$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori*⁴ less than $3\frac{1}{7}$ times the diameter AB .

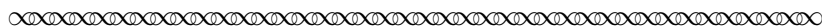


- (e) Archimedes replaced the denominator $4673\frac{1}{2}$ with $4672\frac{1}{2}$. Why? Is this a legitimate step mathematically?
- (f) Compare a decimal approximation of $3\frac{1}{7}$ with a calculator value for π . Express the error of the estimate as a percentage of π .

3 The Inscribed Polygons

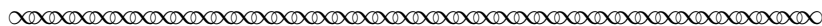
Archimedes then used inscribed regular polygons to establish a lower bound for the ratio of the circumference of a circle to its diameter. This process relied not only on the angle bisector theorem but also on additional properties of proportions. As we will see, Archimedes used triangles based not on a central angle (vertex at the center of the circle) but on an inscribed angle (vertex on the circle itself). He also used similar triangles in a creative and clever way.

We begin by reviewing properties of proportions as presented in Book V of Euclid’s *Elements*.



If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all antecedents be to all the consequents. (V. 12)

If four magnitudes be proportional, they will also be proportional *alternando* (V. 16)⁵



Task 9

Properties of Proportions. Let us explore those Euclidean statements in today’s language of symbolic algebra.

- (a) Proposition V.12 states that if $a, b, c,$ and d are numbers such that $a : b = c : d,$ then

$$a : b = (a + c) : (b + d).$$

Verify the statement for $a = 6, b = 9, c = 8,$ and $d = 12.$

- (b) Proposition V.16 states that if $a, b, c,$ and d are numbers such that $a : b = c : d,$ then

$$a : c = b : d.$$

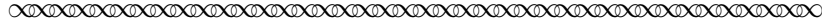
Verify the statement for $a = 6, b = 9, c = 8,$ and $d = 12.$

- (c) Verify Euclid V. 12 and V. 16 using your own values for $a, b, c,$ and $d.$

⁴Meaning “from the stronger” and is used when a conclusion follows even more strongly from an established fact. In other words, if the weaker thing is true, then certainly this stronger thing must be true also.

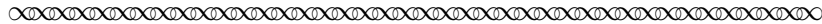
⁵Latin for “by alternation,” meaning that both ratios are true because they express the same proportional relationship in alternated order.

Archimedes began the lower bound process by inscribing a regular hexagon in a circle, as shown in **Figure 8**.



II. Next let AB be the diameter of a circle, and let AC , meeting the circle in C , make the angle CAB equal to one-third of a right angle. Join BC . Then

$$AC : CB [= \sqrt{3} : 1] < 1351 : 780.$$



Task 10

The Inscribed Hexagon. Let us work through the details of the lower bound journey's launch.

- (a) Consider a regular hexagon inscribed in a circle (**Figure 8**). What kind of a triangle is triangle ABC ? How do you know?
- (b) Using modern square root notation, what is the exact ratio of the longer leg AC to the shorter leg BC ? Hypotenuse AB to BC ? Again, note that Heath added the text in square brackets.
- (c) Archimedes approximated the ratio of $AC : CB$ as $1351 : 780$. Use a calculator to determine how close this ratio is to the exact ratio.
- (d) Note the direction of the inequality for $AC : CB < 1351 : 780$. Why do you think Archimedes started with an estimate for $AC : CB$ that was *greater than* its true value? As before, inverting the inequality may be helpful.
- (e) Use this hexagon to make a lower bound approximation for π .

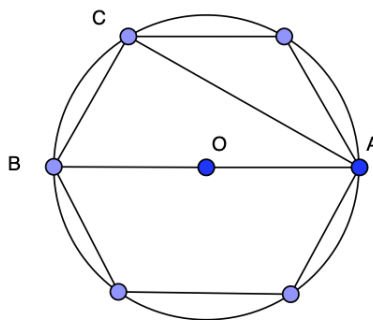
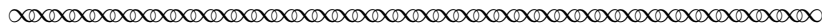


Figure 8: Inscribed Hexagon

Archimedes then bisected the inscribed angle BAC .



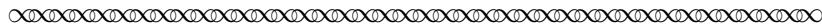
First, let AD bisect the angle BAC and meet BC in d and the circle in D . Join BD . Then

$$\begin{aligned}\angle BAD &= \angle dAC \\ &= \angle dBD,\end{aligned}$$

and the angles at D, C , are both right angles.

It follows that the triangles $ADB, [ACd], BDd$ are similar. Therefore

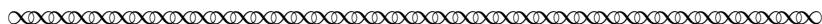
$$\begin{aligned}AD : DB &= BD : Dd \\ &[= AC : Cd] \\ &= AB : Bd && \text{(Euclid VI.3)} \\ &= AB + AC : Bd + Cd \\ &= AB + AC : BC.\end{aligned}$$



Task 11

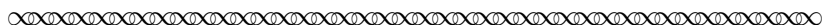
The First Bisection. By bisecting this angle, Archimedes went from an inscribed hexagon to an inscribed 12-gon, which increased the accuracy of the lower bound estimate for π .

- (a) Draw the angle bisector of angle BAC meeting the circle at point D and BC at point d (per Archimedes' original notation). Draw DB (**Figure 9**). How do we know that BD is the side of a regular inscribed 12-gon?
- (b) Why does $\angle dBD = \angle dAC$? Why are triangles ADB, ACd , and BDd similar?
- (c) Explain why $AD : DB = BD : Dd = AC : Cd$.
- (d) Use Euclid VI.3 to show that $AC : AB = Cd : dB$. Then show that $AC : Cd = AB : Bd$.
- (e) Use Euclid V.12 to show that $AD : DB = (BA + AC) : BC$. In words, Archimedes replaced the ratio of the larger leg to the smaller leg of the new triangle ABD with the ratio of the sum of the two largest sides of the old triangle ABC to the smallest side of the old triangle. Explain how this resulting equation is similar to the result in the circumscribed hexagon.



$$\begin{aligned}&[\text{But } AC : CB < 1351 : 780 \text{ from above,} \\ &\text{while } BA : BC = 2 : 1 \\ &= 1560 : 780.] \end{aligned}$$

$$\text{Therefore } AD : DB < 2911 : 780. \tag{1}$$



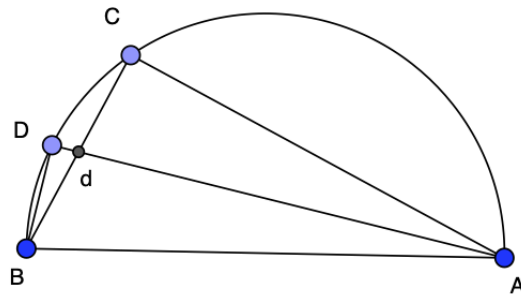
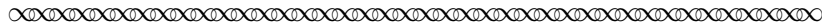


Figure 9: The First Bisection Inscribed

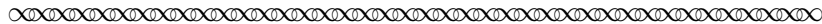
- (f) Verify that $AD : DB < 2911 : 780$. (Note that Archimedes re-numbered the key equations for the inscribed bisections again starting with (1)).

Archimedes next used the Pythagorean Theorem (Euclid I.47) to approximate the ratio of the hypotenuse to the shorter leg of the “new” triangle DAB .



$$\begin{aligned} \text{[Hence } AB^2 : BD^2 &< (2911^2 + 780^2) : 780^2 \\ &= 9082321 : 608400.] \end{aligned}$$

$$\text{Thus } AB : BD < 3013\frac{3}{4} : 780. \tag{2}$$

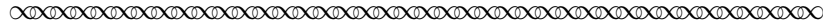


Task 12

Applying the Pythagorean Theorem. By this point, we’re not exactly doubting Archimedes’ ability to correctly use the Pythagorean Theorem. But for our own understanding, let’s check the details of what he wrote above!

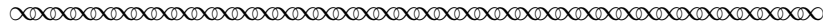
- Work through the first two lines above to show that $AB^2 : BD^2 < 9082321 : 608400$. What does the ratio $9082321 : 608400$ represent in terms of geometric squares built on the hypotenuse and the short side of triangle DAB ?
- Use your calculator to compare $\sqrt{\frac{9082321}{608400}}$ with the decimal value for $\frac{3013\frac{3}{4}}{780}$.
- Notice that 780 is propagating through the calculations. Why?

As with the previous bound, Archimedes realized that iterating the bisection process would tighten the bound.



Secondly, let AE bisect the angle BAD , meeting the circle in E ; and let BE be joined. Then we prove, in the same way as before, that

$$\begin{aligned}
 AE : EB & [= BA + AD : BD \\
 & < (3013\frac{3}{4} + 2911) : 780, \quad \text{by (1) and (2)} \\
 & < 5924\frac{3}{4} : 780 \\
 & < 5924\frac{3}{4} \times \frac{4}{13} : 780 \times \frac{4}{13} \\
 & < 1823 : 240.
 \end{aligned}$$



Task 13

The Second Bisection. Once again, it is worth working through this second iteration ourselves.

- (a) Draw the bisector of $\angle DAB$ meeting the circle at E , and draw BE .
- (b) Explain why $AE : EB = (BA + AD) : BD$. Use this to show that $AE : EB < 5924\frac{3}{4} : 780$.
- (c) Why do you think Archimedes “rescaled” the ratio $5924\frac{3}{4} : 780$ by a factor of $\frac{4}{13}$?
- (d) Apply the Pythagorean Theorem (Euclid I.47) to triangle EAB to show that $AB^2 : BE^2 < 3380929 : 57600$.
- (e) Archimedes claimed $\sqrt{3380929} < 1838\frac{9}{11}$, so that $AB : BE < 1838\frac{9}{11} : 240$. Using your calculator, compare $\sqrt{3380929}$ with $1838\frac{9}{11}$. What if Archimedes had used $1838\frac{8}{11}$ for the approximation to $\sqrt{3380929}$?

Task 14

The Third and Fourth Bisections

As before, Archimedes continued this iterative process for two more bisections, this time to arrive at $\frac{AB}{GB} < \frac{2017\frac{1}{4}}{66}$. **Figure 10** shows all four inscribed angle bisections in one figure. This figure was included in the original Archimedes manuscript and gave an accurate depiction of the ratios of the side lengths for the inscribed polygons.

- (a) Complete the table on the next page that summarizes the results of all four bisections. The first two rows are already filled in, as is the last column regarding the ratio of the hypotenuse to side. As in **Table 1**, the sides of the triangles shown should be interpreted as ratios rather than absolute lengths.
- (b) Explain how the denominator 240 in the third bisection becomes 66 in the fourth bisection, and why Archimedes did this.

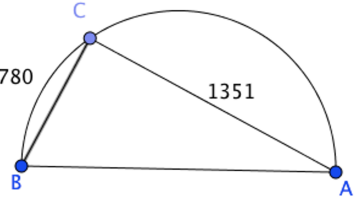
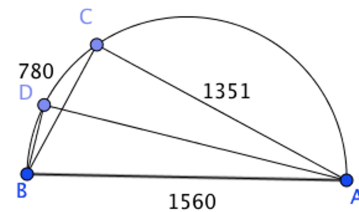
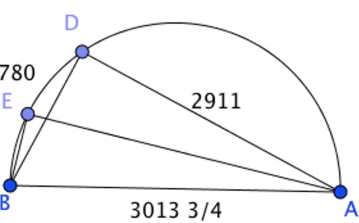
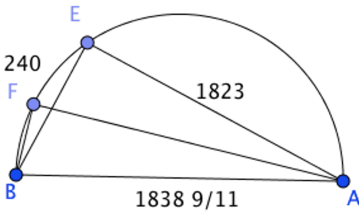
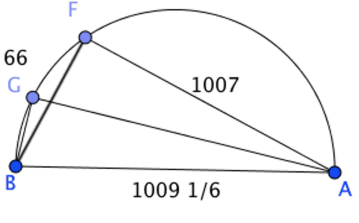
| Ratios of the Sides of the Triangle as Part of the Inscribed Polygon of n sides | New Tri. | Old Tri. | Replacement Equation | New Ratio of Sides | New Ratio of Hyp. to Side |
|---|-----------------|-----------------|------------------------------------|---|--|
|  <p>Starting Triangle ($n = 6$)</p> | $\triangle CBA$ | None | None | $\frac{AC}{CB} < \frac{1351}{780}$ | $AB : CB$ $= 1560 : 780$ $= 2 : 1$ |
|  <p>First Bisection ($n = 12$)</p> | $\triangle DBA$ | $\triangle CBA$ | $\frac{AD}{DB} = \frac{CA+AB}{CB}$ | $\frac{AD}{DB} < \frac{1351+1560}{780}$ $\frac{AD}{DB} < \frac{2911}{780}$ | $AB : DB$ $< 3013\frac{3}{4} : 780$ |
|  <p>Second Bisection ($n = 24$)</p> | | | | $\frac{AE}{EB} < \frac{2911+3013\frac{3}{4}}{780}$ $\frac{AE}{EB} <$ | $AB : EB$ $< 1838\frac{9}{11} : 240$ |
|  <p>Third Bisection ($n = 48$)</p> | | | | $\frac{AF}{FB} < \frac{1823+1838\frac{9}{11}}{240}$ $\frac{AF}{FB} <$ | $AB : FB$ $< 1009\frac{1}{6} : 240$ |
|  <p>Fourth Bisection ($n = 96$)</p> | | | | $\frac{AG}{GB} < \frac{1007+1009\frac{1}{6}}{66}$ $\frac{AG}{GB} <$ | $AB : GB$ $< 2017\frac{1}{4} : 66$ |

Table 2: All Angle Bisections for the Inscribed n -gons

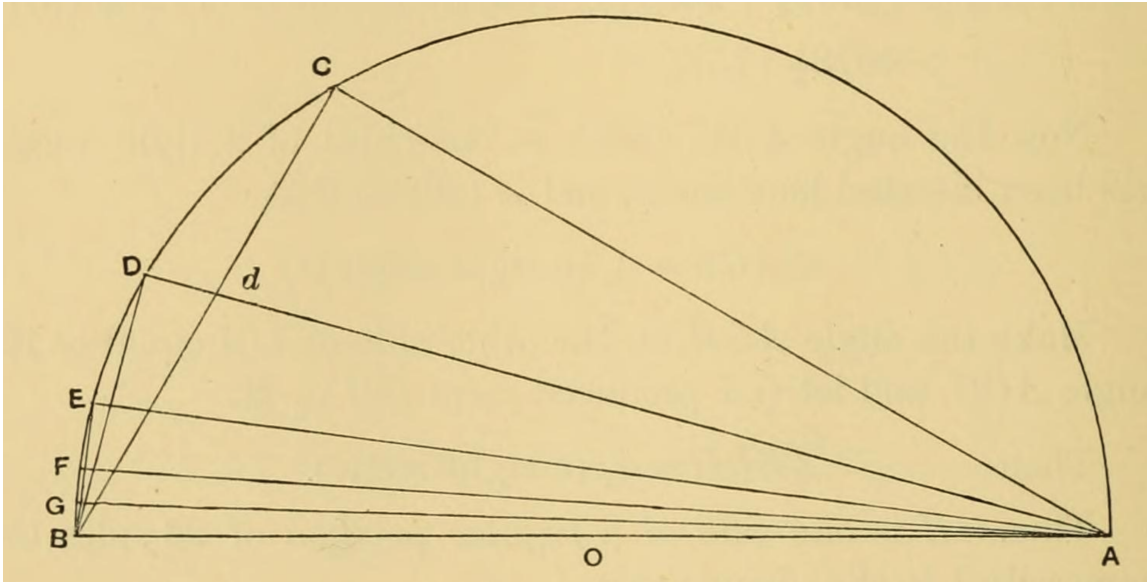
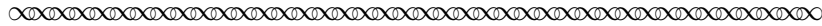


Figure 10: All Inscribed Angle Bisections (Reprinted from (Heath, 1897))

At last, Archimedes extracted the relevant information about π from his inequalities.



$$\begin{aligned} \text{Therefore, } AB : GB &< 2017\frac{1}{4} : 66 \\ \text{whence } BG : AB &> 66 : 2017\frac{1}{4}. \end{aligned} \tag{7}$$

[Now the angle BAG , which is the result of the fourth bisection of the angle BAC , or one-third of a right angle, is equal to one-forty-eighth of a right angle.

Thus the angle subtended by BG at the centre is

$$1/24 \text{ (a right angle).}]$$

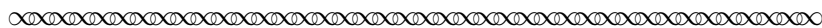
Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

$$\begin{aligned} \text{(perimeter of polygon) : } [AB &> 96 \times 66 : 2017\frac{1}{4}] \\ &> 6336 : 2017\frac{1}{4}. \end{aligned}$$

$$\text{And } \frac{6336}{2017\frac{1}{4}} > 3\frac{10}{71}.$$

Much more then is the circumference of the circle greater than $3\frac{10}{71}$ times the diameter.



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Notes to Instructors

Overview: Topics, Goals, and Task Commentary

The ratio of the circumference of a circle to its diameter is one of the most important numbers in all of mathematics. Although this ratio, commonly notated as π , is used extensively throughout the secondary and post-secondary mathematics curriculum, its origin is not always known to students. This Primary Source Project (PSP) aims to give students the opportunity to explore Archimedes' historical approach in developing one of the earliest approximations to π using an iterative process. This PSP traces the key steps in Archimedes' *Measurement of a Circle* by approximating the side lengths of circumscribed and inscribed regular polygons about and within the circle using the mathematical tools that Archimedes had at his disposal at the time of writing (ca. 180 BCE).

This PSP has two goals: (1) for students to see how the concept of proportion based on similar triangles in Euclidean geometry was leveraged in a powerful way to compute approximations that were incredibly accurate; and (2) to give students an awareness of, and perhaps, appreciation for, the mathematical sophistication of the ancient mathematicians.

Student Prerequisites and Classroom Implementation

In theory, this PSP should require no more background than a secondary-level course in Euclidean geometry. However, our experience was that there is also a mathematical sophistication necessary on the part of the student in order to follow Archimedes' steps which were often stated without explanation. In particular, students often required assistance in setting up the correct correspondence in triangle similarity and expressing the subsequent proportions accurately. Thus, we found that this lesson worked well in an upper-division or first-year graduate course in mathematics history, geometry, and/or mathematical writing.

As with many PSPs, classroom implementation is perhaps most effectively accomplished with a combination of individually assigned work, small-group discussion, and direct instruction/whole group discussion. It may be especially helpful to “walk” students through the triangle similarities and subsequent proportions for both the circumscribed and inscribed polygons in Tasks 3-4 and 10-11.

Archimedes' Use of Ratios of the Sides of a Triangle

Archimedes always referred to the ratio of side lengths rather than as absolute lengths. For example, in **Figure 5**, triangle AOC has side length ratios of $AC : AO : CO$ of 153 : 265 : 306. This does not mean that the lengths of AC , AO , and OC are 153, 265, and 306, respectively.

As a way to help students see the “big picture” for the bisections for both circumscribed and inscribed polygons, we created **Tables 1** and **2**, with the triangles shown with the ratios of their sides. In our original lesson we did not have either table as part of the lesson. However, when we field-tested this lesson for the first time with an actual class, students had a very difficult time keeping track of the “old” and “new” triangles and side length ratios for each iteration, resulting in both frustration and disengagement. It was clear that a coherent visual aid would be necessary – perhaps even critical – for students to be able to follow Archimedes' transition from one triangle to the next following the angle bisection. Thus, we created **Tables 1** and **2** to, hopefully, help students see the effect of the angle bisection and to keep track of the new side ratios in an organized way.

Using the tables in our second field-test lesson seemed to greatly increase students' understanding of the bisection process as well as energize their participation in the tasks. To help clarify the interpretation of the labeled figures in the tables, we included the following note preceding **Table 1** in Task 6.

Important note: In **Table 1**, the labeling of the side “lengths” of the triangles in the first column needs proper interpretation. As we have observed, and as we have done throughout this lesson, Archimedes referred to side length ratios rather than absolute lengths. So, in the first triangle picture, for example, rather than say “the length of AO is 265 and the length of AC is 153,” we should say “the ratio of AO to AC is 265 to 153.” Moreover, a triangle with side length ratios 153 : 251 : 306 is not a 30-60-90 degree triangle (the true ratios are, of course, $1 : \sqrt{3} : 2$). With these two observations in mind, we have found that the pictures are useful in visualizing the bisection process for each iteration, as well as linking the ratios of the sides based on the “old” triangle with the ratio of sides based on the “new” triangle.

A similar note appears immediately preceding **Table 2**.

Numeration and Arithmetic in Archimedes

A theme that was brought up a few times throughout the PSP was the idea of if this work would have been any different if Archimedes was representing his numbers as decimal expansions. This is an example of a question that would really only arise from teaching with a PSP. But a natural follow-up question would be: how *was* Archimedes representing his numbers? And, how was he doing his calculations, including arriving at his (shockingly accurate!) approximations for square roots? The interested reader may consult Chapter 4: Arithmetic in Archimedes in the 2002 Dover Edition of Heath's translation and commentary *The Works of Archimedes* (Heath, 2002).

Sample Implementation Schedule (based on a 75-minute class period)

Two separate class periods of at least 75 minutes are needed for this PSP. Ideally, the class sessions would be sequential with a day “off” in between (M W or T Th). Please note that some parts of the tasks involve verifying Archimedes' fractional approximations to square roots using a calculator. These verifications, while most impressive, can be done by students outside of class time and are noted here as being optional.

- **Day 1 Pre-lesson Assignment.** Students should read the Introduction section and work through Task 1, using the applet, as well as Task 2 on The Angle Bisector Theorem.
- **Day 1**
 - *Task 2: The Angle Bisector Theorem.* Small group discussion followed by whole class discussion on this important but lesser-known result from Euclid. Please note the use of the parallel postulate in Euc I.29.
 - *Task 3: The Circumscribed Hexagon.* Whole class discussion. Although students will be familiar with the ratios of the sides of a 30-60-90 triangle, the Greek notion of areas of squares constructed on each side may be a new idea.

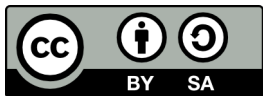
- *Tasks 4 and 5: The First Bisection and Applying the Pythagorean Theorem.* Whole class discussion. These two tasks set the stage for the entire PSP and so it is important that each student be able to work through both of them successfully. It is especially important for students to understand the verbal description of how Archimedes transitioned from each n -gon to the next $2n$ -gon. If desired, small groups might be used in comparing Archimedes’ rational estimates for the square roots.
 - *Task 6: The Second Bisection.* Small group discussion. This substantive task presents the key step in students coming to understand Archimedes’ approach with the circumscribed polygons.
 - *Task 7: The Third and Fourth Bisections.* Small group discussion. Rather than going through the details of Archimedes’ steps, this table allows students to focus on the transition from each n -gon to the subsequent $2n$ -gon.
 - *Task 8: Creating an Upper Bound.* Whole group discussion. Based on time constraints and the importance of Archimedes’ lower bound based on the fourth bisection, we recommend that the instructor be actively involved in this discussion.
- **Day 2 Pre-lesson Assignment.** Students should work through *Task 9: Properties of Proportions*. Be sure that each student successfully illustrated Euclid V. 12 and 16 using their own numbers. A short group discussion at the beginning of Day 2 is a good idea to ensure that students understand this material.
- **Day 2**
 - *Task 10: The Inscribed Hexagon.* Small group discussion. Please note that the triangles associated with the inscribed polygons are based on inscribed rather than central angles. It is important to consider why the inscribed angles will create a regular polygon having the desired central angles.
 - *Tasks 11 and 12: The First Bisection and Applying the Pythagorean Theorem.* Whole class or small group discussion. We found the three similar triangles that arise in the inscribed polygons to be a little trickier to understand than those in the circumscribed polygons; some whole class discussion may be helpful. As was the case for the inscribed polygons, it is especially important for students to understand the verbal description of Archimedes’ transition from each n -gon to the next $2n$ -gon. If desired, small groups might be used in comparing Archimedes’ rational estimates for the square roots.
 - *Task 13: The Second Bisection.* Small group discussion. This substantive task again presents the key step in students coming to understand Archimedes’ approach with the inscribed polygons.
 - *Task 14: The Third and Fourth Bisections.* Small group discussion. As before, this table allows students to focus on the transition from each n -gon to the subsequent $2n$ -gon.
 - *Task 15: Creating a Lower Bound.* Small group discussion. This task is the “feather in the cap” for Archimedes’ piece. Students might feel a great sense of accomplishment if they can figure this out with minimal assistance from the instructor.
- **Post-lesson Assignment**
 - Task 16: Reflections.* We would encourage students to write down their thoughts on the lesson and, if time permits, briefly share these either in class or on a discussion board.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching topics related to geometry and/or Greek mathematics. The PSP author name of each is given (together with the content focus). Classroom-ready versions of these projects can be downloaded from <https://digitalcommons.ursinus.edu/triumphs/>.

- *The Exingency of the Euclidean Parallel Postulate and the Pythagorean Theorem*, Jerry Lodder.
- *The Failure of the Euclidean Parallel Postulate and Distance in Hyperbolic Geometry*, Jerry Lodder.
- *A Genetic Context for Understanding the Trigonometric Functions*, Danny Otero.
- *Generating Pythagorean Triples: A Gnomonic Exploration*, Janet Heine Barnett.

Acknowledgements



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